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A handy chart to navigate a *small* part of Ayoub's Thesis on Grothendieck's six operations in the motivic stable homotopy categories

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Abstract

This is intended to be a handy chart to navigate the basic part of Ayoub's interesting, but very lengty, thesis on Grothendieck's six operatiosn in the motivic stable homotopy categories.

1 Introduction

In recent years mathematics of various disciplines are intimately interacting each other. For instance, taking a glance at the articles in [Toric], we are convinced that transformation group theorists can not get away from algebraic geometry any more. This suggests, for instance, transformation group theorists may have to face Grothendieck's SGA [SGA 4, SGA 4+1/2, SGA 6]. Of course, the étale cohomology is developped there, but, more fundamentally, the so called "Grothendieck's six operations" $(\mathsf{R}f_*, \mathsf{L}f^*, \mathsf{R}f_!, \mathsf{R}f^!, \otimes^{\mathsf{L}}, \mathsf{R}\,\mathrm{Hom})$.

For a scheme X of finite type over a noetherian base scheme S. let $D(X)$ be the (appropriately defined) derived category of l-adic sheaves on X. Then, for (reasonable) X and $K \in D(S)$, we have the following expressions of the usual cohomology, the cohomology with compact coefficients, the usual homology, and the Borel-Moore homology:

$$\begin{aligned} H^n(X,K) &= \mathrm{Hom}_{D(S)}\left(\mathbf{1}, \mathsf{R}p_*\mathsf{L}p^*K[n]\right) & H_n(X,K) &= \mathrm{Hom}_{D(S)}\left(\mathbf{1}, \mathsf{R}p_!\mathsf{R}p^!K[-n]\right) \\ H^n_c(X,K) &= \mathrm{Hom}_{D(S)}\left(\mathbf{1}, \mathsf{R}p_!\mathsf{L}p^*K[n]\right) & H^{BM}_n(X,K) &= \mathrm{Hom}_{D(S)}\left(\mathbf{1}, \mathsf{R}p_*\mathsf{R}p^!K[-n]\right) \end{aligned} \tag{1}$$

Here $\mathbf{1}$ is the unital object in the tensor category $D(S)$ and, in general. for an appropriate $f: X \to Y$,

$$\begin{aligned} \mathsf{R}f_*, \mathsf{R}f_! &: D(X) \to D(Y) \\ \mathsf{L}f^*, \mathsf{R}f^! &: D(Y) \to D(X) \end{aligned} \tag{2}$$

(See [DV, 1,1,3]. For a related motivic story, consult [L, IV, 2.2.2] for instance.)

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The expressions (1) alone would manifestly indicate the Grothendieck's six operations between triangulated categories are more fundamental than (co)homology, even without knowing the rich applications of six operations as was developped in [SGA 4, SGA 4+1/2, SGA 6]. Of course, many topologists and geometers would say, there are also analogous six operations and (co)homology expressions in the context real analytic geometry [KS]. Anyway, transformation group theorists would easy realize an importance of understanding the Grothendieck's six operations, or at least those four operations (1)(2).

On the other hand, speaking of algebraic geometry, what homotopy theorists nowadays first think of would be the motoivic homotopy theory developped by Morel and Voeodsky [MV]. The subsequent work [J][Ho2] constructed its stable versions as certain stable model categories [Ho1][Hi] whose homotopy category become a triangulated category, so called the motivic stable homotopy category. Having a triangulated category at hand, it was natural to seek analogues of Grothendieck's six operations in this framework. In fact, Voevodsky outlined [DV] how to realize this goal at the category of motivic symmetric spectra (which is the monoidal stable model category whose homtopy category is nothing but the unital tensor triangulated category, called the motivic stable homotopy cateogry) in terms of "cross functor"which is described in the 2-category, 2-functor languages. Actually, the relevance of the 2-category, 2-functor languages appears natural, even from naive observations:

- some interesting functors are only functorial at the homotopy category:

$$F(gf) \xrightarrow{\simeq} F(g)F(f)$$

- in the Grothendieck six functor formalism, there is a natural transformation

$$f_! \to f_*$$

  for separated morphisms.

It turns out that Voevodsky's approach required somewhat substantial technical detail, and finally achived by Ayoub in his thesis which was publisshed in [A] as two Astérisque volumes of total more than 830 pages. Naturally, the size of these volumes have intimidated those interested, very undortunatel. Actually, Ayoub's thesis is well written and self contained.

Now this paper was originally prepared for the author's own sake to use as a handy guideline chart to read certain small part of Ayoub's long thesis [A]. Although I used an expression "a small part" above, this part cotains Ayoub's construction of the motivic stable homotopy analogue of Grothendieck's four functors, which spans several hundread pages in Ayoub's thesis. Such is the case, this paper is more or less a rearranged list of rerelvanet statements stated in the small part of Ayoub's thesis, except some detailed explanation are supplied to certain parts. Such is the case, basically there is no originality in this write up, but possibly many mistakes resulting solely from the author's lack of ability. Furthermore, this papar can not be read independently, without Ayoub's thesis at hand, because this paper is designed to assist readers of Ayoub's thesis. Of course, I should really apoplogize readers for this. However, considering impressive applications of Ayoub's huge volumes found in the Feynman motives of Marcolli and her collaborators [Mar] and Cisinski's cdh descent theorem for homotopy invariant K-theory [C] to state a few, I hope (at least certain part of) this paper, or "chart" , would be of some use for some interested readers.

Having said this, it might be still not convincing for most transformation group theorits to be motivated to read Ayoub's thesis. For those folks, let me suggest to practice the following:

- Take a look at Cisinski's beautiful argument in [C, §3], where the cdh descent property of the homotopy invariant K-theory is derived so instantaneously from the localisation, the smooth base change, and the proper base change, all of which were shown in Ayoub's thesis.
- To get some topological insight about localisation and base change, consult [I, II(6.11)(6.13); VII,2.6], and possibly [Mil, II §3 p.76; VI Cor.2.3, Th.4.1].
- To understand the categorical origin of the base change is adjunction, consult [DV,].
- Now, you may be well motoviated to read Ayoub's thesis.

Actually, I myself was so interested in Ayoub's thesis through my perticipation in the Yatsugatake workshop 2016, Descent for algebraic K-theory, where the Ayoub's thesis and its application to Cisinski's cdh descent theorem of homotopy invariant K-theory. Now this chart is organized as follows:

1. Introduction
2. A glimpse of the framework of Ayoub's construction
3. Constructions of $SH(X), f_\sharp \dashv f^* \dashv f_*$
4. $\mathbb{SH}_{\mathfrak{M}} : \mathbf{Dia\,Sch} \to \mathfrak{MonoTR}$ is a stable homotopy algebraic derivataeur
5. Construction of the adjunction $f_! : Sh(X) \rightleftarrows Sh(Y) : f^!$
6. The proper base change theorem

Unfortunately, the last section about the proper base change is really sketchy, but Cisinski's proof of the cdh descent of the homotopy K-theory only requires the proper base change for closed immersions, which was reviewed in 5.4.

After the preliminary version of this chart was written, I noticed the papers of Hoyois [Hoy1][Hoy2]. In these papers, Hoyois exploited Lurie's technique of ∞-category to generalize Ayoub's construction of Grothendieck operations [Hoy1] and generalized Cisinski's cdh descent theorem for homotopy invariant K-theory [Hoy2].

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2 A glimpse of the framework of Ayoub's construction

In this section, we shall summarize the basic framework ofAyoub's construction by simply quoting corresponding statements from Ayoub's thesis. For brevity and conciseness, we shall not review the definitions and notations in advance. We may review only some of them later.

2.1 Basic idea

We work in a (appropriately defined) categoory of S-schemes $\mathbf{Sch}/S$ as follows:

[Ayoub, 1.3.5, p.53; Lem.1.3.9, P.54] [FGA, p.126]

- Let $\mathbf{Sch}/S$ be either one of the followings:
  - When S is general, we can take $\mathbf{Sch}/S$ to be the category of S-schemes, which are quasi-projective in the sense of Hartshorne's textbook: S-schemes of finite presentation which admits an immersion in $\mathbb{P}^n_S$ for sufficiently large n.
  - If S admits an ample family of line bundles, we can take $\mathbf{Sch}/S$ to be the category of S-schemes, which are quasi-projective in the sense of Grothendieck's EGA2: S-schemes of finite presentation which admits an immersion in $\mathbb{P}(\mathcal{M})$ with $\mathcal{M}$ a coherent $\mathcal{O}_S$ module, which is not necessarily locally free.

  Then, whichever choice we adopt, any morphism $f : X \to Y$ in $\mathbf{Sch}/S$ factorizes as follows:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}(\mathcal{L}) \arrow[d, "p"] \\ & Y \end{tikzcd}$$

  with i an immersion, $\mathcal{L}$ a locally free $\mathcal{O}_Y$-module and p the canonical projection. (*This is proved in SGA 6 by Illusie.*)

- Following Altman-Kleiman, in [FGA, p.126], a morphism $X \to S$ of Noetherian schemes is called strongly projective (respectively, strongly quasi-projective) if there exists a vector bundle E on S together with a closed embedding (resp. a locally closed embedding) $X \subset \mathbb{P}(E)$ over S.

For a S-scheme $X \to S$, Morel-Voevodsky style motiic stable homotopy category $SH(X)$ is constructed as a triangulated category:

$$\begin{aligned} (\mathbf{Sch}/S)^{op} &\to \mathfrak{TR} \\ (X \to S) &\mapsto SH(X) \end{aligned} \tag{3}$$

We hope this correspondence should behave appropriately with repsect to appropriate morphisms $f : X \to Y$ in $\mathbf{Sch}/S$ to reflect at least four of the Grothendieck six operations:

$$f^*, \quad f_*, \quad f^!, \quad f_!$$

which enjoy some properties.

It turns out that the "exclaimation maps" $f^!, f_!$ are harder to construct. So, we start with constructing an auxiliary map

$$f_\sharp : SH(X) \to SH(Y)$$

for a smooth S-morphism $f : X \to Y$. More precisely, we shall construct a refined version of the naively defined functor (3) so as to become a stable homotopy 2-functor, in the following sense:

[Deligne-Voevodsky,2.1.2.1, p.38][Ayoub, p.54, Def.1.4.1. p.55]

A 2-functor

$$\mathbf{H}^* : (\mathbf{Sch}/S)^{op} \to \mathfrak{TR}$$

is called stable homotopy 2-functor if the following 6 axioms are satisfied:

1. (Trivial tiangulated category) $\mathbf{H}(\emptyset) = 0$.
2. (right adjoint) For any morphism $f : X \to Y$ in $(\mathbf{Sch}/S)$, the 1-morphism $f^* : \mathsf{H}(Y) \to \mathsf{H}(X)$ admits a right adjoint f_* (i.e. f_* is a triangulated functor which a right adjoint of the functor f^*). Furthermore, for a (not necessarily closed) immersion i, the counit 2-morphism $i^*i_* \to \mathrm{Id}$ is a 2-isomorphism.
3. (left adjoint (smooth base change)) If $f : X \to Y$ is a smooth S-morphism in $(\mathbf{Sch}/S)$, the 1-morphism f^* admits a left adjoint $f_\sharp$. Furthermore, for any caresian diagram

   $$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

   the exchange 2-morphism (which shall be defined in the sub-section 1.4.5):

   $$f'_\sharp g'^* \to g^* f_\sharp$$

   is a 2-isomorphism.
4. (Locality) Let $j : U \to X$ be an open immersion in $\mathbf{Sch}/S$ and $i : Z \to X$ is a complementary closed immersion in $\mathbf{Sch}/S$. Then the pair (j^*, i^*) is conservative..
5. (Invariance by homotopy) If $p : \mathbb{A}^1_X \to X$ the canonical projection, the unit 2-morphism :

   $$\mathrm{Id} \to p_* p^*$$

   is a 2-isomorphism.
6. (stability) If s is the zero section of the canonical projecton $p : \mathbb{A}^1_X \to X$, the endofunctor $p_\sharp s_*$ of $\mathsf{H}(X)$ is an equivalence of categories.

Actually, as the following observation indicates, a stable homotopy 2-functor gives us a desired four of the Grothendieck six operations. (Here, topologists might feel good to see that the key to proceed from $f_\sharp$ to the "exclaimation maps" $f^!, f_!$ originates in topology, i.e. the Thom element $\mathbf{Th}$.)

[Ayoub, Scholium.1.4.2. p.55] [Deligne-Voevodsky, p.39]

Suppose we are given a stable homotopy 2-functor

$$\mathbf{H}^* : (\mathbf{Sch}/S)^{op} \to \mathfrak{TR}$$

1. There exist :
   - a contravariant 2-functor
   $$\mathbf{H}^! : (\mathbf{Sch}/S) \to \mathfrak{TR},$$
   - a covariant 2-functor
   $$\mathbf{H}_* : (\mathbf{Sch}/S) \to \mathfrak{TR},$$
   which is a global right adjoint of $\mathbf{H}^*$,
   - a covariant 2-functor
   $$\mathbf{H}_! : (\mathbf{Sch}/S) \to \mathfrak{TR},$$
   which is a global left adjoint of $\mathbf{H}^!$,
   - a crossed functor structure on the quadruplet $(H^*, H_*, H_!, H^!)$ relative to the classes of cartesian squares of $(\mathbf{Sch}/S)$.
2. For any quasi-projective S-scheme X and locally free coherent $\mathcal{O}_X$-module $\mathcal{M}$, there exists an autoexicalence $\mathbf{Th}(\mathcal{M})$ of the restriction of the preceeding cross functor of the cteogry $(\mathbf{Sch}/X)$. If the $\mathcal{O}_X$-module $\mathcal{M}$ is inserted into a short exact sequence:
$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{L} \to 0$$
we get an isomorphism of autoequivalences of cross functors :
$$\mathbf{Th}(\mathcal{M}) \xrightarrow{\sim} \mathbf{Th}(\mathcal{L}) \circ \mathbf{Th}(\mathcal{N})$$
3. Let $f : X \to Y$ be a smooth S-morphism. Denote by Ω_f the locally freee $\mathcal{O}_X$-module of relative differentials. Then there exist 2-isomorphisms:
   - $f_! \xrightarrow{\sim} f_\sharp \mathbf{Th}^{-1}(\Omega_f)$
   - $f^! \xrightarrow{\sim} \mathbf{Th}(\Omega_f) f^*$

   with $\mathbf{Th}^{-1}(\Omega_f)$ the inverse equivalence of $\mathbf{Th}(\Omega_f)$.
4. For any morphism $f : X \to Y$ in $(\mathbf{Sch}/S)$ there exists a 2-morphism : $f_! \to f_*$. When f is projective, this 2-morphism is invertible.
5. We have the base change theorem for a projective morphism , i.e. for any cartesian square:
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$
in $(\mathbf{Sch}/S)$ with f a projective S-morphism, the exchange 2-morphisms:
   - $g^* f_* \xrightarrow{\sim} f'_* g'^*$.
   - $f'_! g'^! \xrightarrow{\sim} g^! f_!$.

   are invertible.

2.2 Slight generalization

Actually, Ayoub formulated and proved a slightly modified version in the diagram scheme setting. For this purpose, let us first quote the basic definition of $\mathbf{Dia}\,\mathbf{Sch}/S$ for this purpose:

[Ayoub, 2.1.2, p.189; 2.4.1. Def.2.4.2. Def.2.4.4. p.310; p.322]

- Fix a full subcategory **Dia** of the strict 2 category of small categories, satisfying:
  - contains $\emptyset, \mathbf{e} := \{\bullet\}, \underline{\mathbf{1}} := \{0 \to 1\}$.
  - underlying 1 category of **Dia** is closed w.r.t. finite coproducts and fiber products.
  - for any functor $u : A \to B$ in **Dia** and any $b \in \mathrm{Ob}(B)$, both A/b and $b\backslash A$ are in **Dia**.
- The 2-category $\mathbf{Dia\,Sch}/S$ of diagrams of quasi-projective S-schemes is defined as follows:
  - An object of $\mathbf{Dia\,Sch}/S$ is a pair $(\mathcal{F},\mathcal{I})$ with $\mathcal{I} \in \mathrm{Ob}(\mathbf{Dia})$ and $\mathcal{F} : \mathcal{I} \to \mathbf{Sch}/S$ a functor.
  - A 1-morphism from $(\mathcal{G},\mathcal{J})$ to $(\mathcal{F},\mathcal{I})$ in $\mathbf{Dia\,Sch}/S$ is given by a functor $\alpha : \mathcal{J} \to \mathcal{I}$ and a natural transformation $f : \mathcal{G} \to \mathcal{F} \circ \alpha$:

$$\begin{tikzcd} \mathcal{J} \arrow[dr, "\mathcal{G}"] \arrow[dd, "\alpha"'] & \\ \arrow[r, phantom, "\not\Downarrow_f"] & \mathbf{Sch}/S \\ \mathcal{I} \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

  - Given two 1-morphisms $(f,\alpha),(f',\alpha')$ from $(\mathcal{G},\mathcal{J})$ to $(\mathcal{F},\mathcal{I})$, a 2-morphism from (f,α) to (f',α') in $\mathbf{Dia\,Sch}/S$ is the data of a natural transformation $t : \alpha \to \alpha'$ such that the following square commutes:

$$\begin{tikzcd} \mathcal{G} \arrow[r, equal] \arrow[d, "f"'] & \mathcal{G} \arrow[d, "f'"] \\ \mathcal{F}\circ\alpha \arrow[r, "\mathcal{F}\circ t"] & \mathcal{F}\circ\alpha' \end{tikzcd}$$

  The 2-category $\mathbf{Dia\,Sch}/S$ defined in this way is a strict 2-category.
- Consider the following subcategories of $\mathbf{Dia\,Sch}/S$:

  $(\mathbf{Dia\,Sch}/S)^{\mathrm{Cart}}$ the same object as $\mathbf{Dia\,Sch}/S$ and for morphisms those wchi are of the form:

$$f : (\mathcal{Y},\mathcal{E}) \to (\mathcal{X},\mathcal{E})$$

  with f a cartesin morphism of $\mathcal{E}$-schemes.

  $(\mathbf{Dia\,Sch}/S)^{\mathrm{LissCart}}$ the same as the first, plus f smooth.

  $(\mathbf{Dia\,Sch}/S)^{\mathrm{ImmCart}}$ the same as the first, plus f a closed immersion.

  $(\mathbf{Dia\,Sch}/S)^{\mathrm{Cart}'}$ the same as the first, plus the $\mathcal{X}$-scheme $\mathcal{Y}$ strongly quasi-projective.

Then we shall construct an even more refined diagram version of (3) so as to become a stable homotopy algebraic derivateur in the following sense:

[Ayoub, Def.2.4.13, p.313]

An algebraic pre-derivateur

$$\mathbb{D} : \mathbf{Dia\,Sch}\,/S \to \mathfrak{TR},$$

valued in the 2-category of triangluated categories, is called a stable homotopy algebraic derivateur when the following axioms **DerAlg0, DerAlg1, DerAlg2d, DerAlg2g, DerAlg3d, DerAlg3g, DerAlg4, DerAlg5** are satisfied:

DerAlg0 Let $(\mathcal{F},\mathcal{I})$ be a diagram of quasi-prpjective S-schemes. If $\mathcal{I}$ is a discrete category, then the 1-morphisms $i : (\mathcal{F}(i), \mathrm{e}) \to (\mathcal{F},\mathcal{I})$ for $i \in \mathrm{Ob}(\mathcal{I})$ induce an equivalence of categories:

$$\mathbb{D}(\mathcal{F},\mathcal{I}) \xrightarrow{\prod_{i\in \mathrm{Ob}(\mathcal{I})} i^*} \prod_{i\in \mathrm{Ob}(\mathcal{I})} \mathbb{D}(\mathcal{F}(i))$$

DerAlg1 Let $(\mathcal{F},\mathcal{I})$ be a diagram of quasi-projective S-schemes and $\alpha : \mathcal{J} \to \mathcal{I}$ an essentially surjective functor. Then the triangulated functor

$$\alpha^* : \mathbb{D}(\mathcal{F},\mathcal{I}) \to \mathbb{D}(\mathcal{F}\circ\alpha,\mathcal{J})$$

is conservative.

DerAlg2d For any 1-morphism $(f,\alpha) : (\mathcal{F},\mathcal{I}) \to (\mathcal{G},\mathcal{J})$ of $\mathbf{Dia\,Sch}\,/S$, the functor $(f,\alpha)^*$ admits a right adjoint $(f,\alpha)_*$.

DerAlg2g For any 1-morphism $(f,\alpha) : (\mathcal{F},\mathcal{I}) \to (\mathcal{G},\mathcal{J})$ of $\mathbf{Dia\,Sch}\,/S$, which is smooth argument by argument, the functor $(f,\alpha)^*$ admits a left adjoint $(f,\alpha)_\sharp$.

Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism of $\mathcal{I}$-diagrams of quasi-projective S-schemes and $\alpha : \mathcal{J} \to \mathcal{I}$ a functor in **Dia**. We have a square:

$$\begin{tikzcd}
(\mathcal{G}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"] \arrow[d, "f|_{\mathcal{J}}"'] & (\mathcal{G},\mathcal{J}) \arrow[d, "f"] \\
(\mathcal{F}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"] & (\mathcal{F},\mathcal{I})
\end{tikzcd}$$

which is commutative (even cartesian) in $\mathbf{Dia\,Sch}\,/S$.

DerAlg3d The exchange 2-morphism $\alpha^* f_* \to (f|_{\mathcal{J}})_* \circ \alpha^*$, associated to the above commutative square, is a 2-isomorphism.

DerAlg3g Suppose f is cartesian and smooth argument by argument. Then the exchange 2-morphism $(f|_{\mathcal{J}})_\sharp \circ \alpha^* \to \alpha^* f_\sharp$ is a 2-isomorphism.

DerAlg4 For any quasi-projective S-scheme X, the 2-functor:

$$\begin{aligned}\mathbb{D}(X,-) : \mathbf{Dia} &\to \mathfrak{TR}\\ \mathcal{I} &\mapsto \mathbb{D}(X,\mathcal{I})\end{aligned}$$

is a triangulated derivateur in the sense of Definition 2.1.34.

DerAlg5 The 2-functor

$$\begin{aligned}\mathbb{D}(-,\mathrm{e}) : \mathbf{Sch}\,/S &\to \mathfrak{TR}\\ X(\text{quasi-projective}) &\mapsto \mathbb{D}(X,\mathrm{e})\end{aligned}$$

is a stable homotopy 2-functor.

Actually, properties in $\mathbf{Sch}\,/S$ and $\mathbf{Dia\,Sch}\,/S$ are very similar and analogous, as is listed in [Ayoub, p.322] as follows:

It turns out that, once constructed, verifications of axioms except **DerAlg5** is easier [Ayoub, Th.4.5.30, p.542]. The remaining **DerAlg5** is nothing but a statement about a stable homotopy 2-functor, which

$\mathsf{H}^* : \mathbf{Sch}/S \to \mathfrak{TR}$ $\mathsf{H}_* : \mathbf{Sch}/S \to \mathfrak{TR}$	$\mathbb{H}^* : \mathbf{Dia\,Sch}/S \to \mathfrak{TR}$ $\mathbb{H}_* : \mathbf{Dia\,Sch}/S \to \mathfrak{TR}$
${}^{\mathrm{Liss}}\mathsf{H}_\sharp : (\mathbf{Sch}/S)^{\mathrm{Liss}} \to \mathfrak{TR}$ ${}^{\mathrm{Liss}}\mathsf{H}^* : (\mathbf{Sch}/S)^{\mathrm{Liss}} \to \mathfrak{TR}$	${}^{\mathrm{LissCart}}\mathbb{H}_\sharp : (\mathbf{Dia\,Sch}/S)^{\mathrm{LissCart}} \to \mathfrak{TR}$ ${}^{\mathrm{LissCart}}\mathbb{H}^* : (\mathbf{Dia\,Sch}/S)^{\mathrm{LissCart}} \to \mathfrak{TR}$
${}^{\mathrm{Liss}}\mathsf{H}_* : (\mathbf{Sch}/S)^{\mathrm{Imm}} \to \mathfrak{TR}$ ${}^{\mathrm{Liss}}\mathsf{H}^! : (\mathbf{Sch}/S)^{\mathrm{Imm}} \to \mathfrak{TR}$	${}^{\mathrm{LissCart}}\mathbb{H}_* : (\mathbf{Dia\,Sch}/S)^{\mathrm{ImmCart}} \to \mathfrak{TR}$ ${}^{\mathrm{LissCart}}\mathbb{H}^! : (\mathbf{Dia\,Sch}/S)^{\mathrm{ImmCart}} \to \mathfrak{TR}$
$\mathsf{H}_! : \mathbf{Sch}/S \to \mathfrak{TR}$ $\mathsf{H}^! : \mathbf{Sch}/S \to \mathfrak{TR}$	${}^{\mathrm{Cart}}\mathbb{H}_! : (\mathbf{Dia\,Sch}/S)^{\mathrm{Cart}'} \to \mathfrak{TR}$ ${}^{\mathrm{Cart}}\mathbb{H}^! : (\mathbf{Dia\,Sch}/S)^{\mathrm{Cart}'} \to \mathfrak{TR}$
locally free $\mathcal{O}_X$-module of finite type $\mathcal{M}, \mathcal{N}, \mathcal{L} \ldots$ etc	locally free and coherent $\mathcal{O}_X$-module of finite type $\mathcal{M}, \mathcal{N}, \mathcal{L} \ldots$ etc

was the ultimate goal in 2.1. Basic idea, quoted as [Ayoub, p.54; Def.1.4.1. p.55].

3 Constructions of $SH(X)$, $f_\sharp \dashv f^* \dashv f_*$

We shall work in the slightly generalized diagram setting as in 2.2, and we shall construct the desired objects and adjunctions $f_\sharp \rightleftarrows f^* \rightleftarrows f_*$ in the following order:

$$\mathbf{Dia\,Sch}/S \ni (\mathcal{F},\mathcal{I}) \rightsquigarrow \mathbf{PreShv}\,(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathcal{M}) \rightsquigarrow \mathrm{M}_T(\mathcal{F},\mathcal{I}) := \mathbf{Spect}^{\Sigma}_{T_{\mathcal{F}},\mathcal{I}}\,(\mathbf{PreShv}\,(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathcal{M})) \\ \rightsquigarrow \mathbb{SH}^T_{\mathcal{M}}(\mathcal{F},\mathcal{I}) := \mathbf{Ho}\,(\mathrm{M}_T(\mathcal{F},\mathcal{I})) \in \mathfrak{TR}$$

In the special case of the constant diagram $\mathcal{I} = \mathbf{e}$ valued at $X \in \mathbf{Sch}/S$,, we shall set

$$\mathbf{SH}^T_{\mathcal{M}}(X) := \mathbb{SH}^T_{\mathcal{M}}(X,\mathbf{e})$$

and we shall further denote it simply by $SH(X)$ under the case we shall be considering (which must be justified).

3.1 Construction of $\mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathcal{M})\,,\,(f,\alpha)_\sharp \dashv (f,\alpha)^* \dashv (f,\alpha)_*$

[Ayoub, 4.5.1, Def.4.5.1, Lem.4.5.2, Prop.4.5.3, p.532]

- Given a diagram $(\mathcal{F},\mathcal{I})$ of S-schemes, We denote $\mathrm{Sm}\,/(\mathcal{F},\mathcal{I})$ the category:
  - an object is of the form
  $$((U \to \mathcal{F}(i)) \in \mathrm{Sm}\,/\mathcal{F}(i), i \in \mathrm{Ob}(\mathcal{I}))\,,$$
  which shall be simply denoted by (U,i).
  - an arrow $(U',i') \to (U,i)$ is a couple $(U' \to U, i' \to i)$ such that the following square commutes:
  $$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{F}'(i') \arrow[r] & \mathcal{F}(i) \end{tikzcd}$$
- A 1-morphism $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ of $\mathbf{Dia\,Sch}\,/S$ yields a canonical factorisation:
$$(\mathcal{G},\mathcal{J}) \xrightarrow{f} (\mathcal{F}\circ\alpha,\mathcal{J}) \xrightarrow{\alpha} (\mathcal{F},\mathcal{I})$$
- f induces a functor
$$\begin{aligned} f = - \times_{\mathcal{F}} \mathcal{G} : \mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}) &\to \mathrm{Sm}\,/(\mathcal{G},\mathcal{J}) \\ ((V \to \mathcal{F}(\alpha(j)))\,, j) &\mapsto \big((V \times_{\mathcal{F}(\alpha(j))} \mathcal{G}(j), j)\big)\,, \end{aligned}$$
which in turn induces an adjunction (c.f. Lem.4.4.44)
$$(f^*, f_*) : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}),\mathfrak{M}) \to \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{G},\mathcal{J}),\mathfrak{M})$$
- α induces a functor
$$\begin{aligned} \overline{\alpha} : \mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}) &\to \mathrm{Sm}\,/(\mathcal{F},\mathcal{I}) \\ ((U \to \mathcal{F}(\alpha(j)))\,, j) &\mapsto ((U \to \mathcal{F}(\alpha(j)))\,, \alpha(j)) \end{aligned}$$
which in turn induces the functor (c.f. Lem.4.4.44)
$$\alpha^* := \overline{\alpha}_* : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathfrak{M}) \to \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}),\mathfrak{M})$$
- Define the functor
$$(f,\alpha)^* := f^* \circ \alpha^* : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathfrak{M}) \to \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{G},\mathcal{J}),\mathfrak{M})$$
- Explicitly, for $H \in \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathfrak{M})$, $(f,\alpha)^*H$ is given by the association:
$$\begin{aligned} (f,\alpha)^*H : \mathrm{Ob}\,(\mathrm{Sm}\,/\mathcal{G}(j)) &\to \mathrm{Ob}\,\mathfrak{M} \\ (V,j) &\mapsto \mathrm{Colim}_{(V \to U \times_{\mathcal{F}(\alpha(j))} \mathcal{G}(j)) \in \mathrm{Ob}(V\backslash(\mathrm{Sm}\,/\mathcal{F}(\alpha(j))))}\, H(U,\alpha(j)) \end{aligned}$$
- The association $(f,\alpha) \mapsto (f,\alpha)^*$ extends naturally to a contravariant 2-functor:
$$\begin{aligned} \mathbf{PreShv}\,(\mathrm{Sm}\,/(-,-),\mathfrak{M}) : \mathbf{Dia\,Sch}\,/S &\to \mathfrak{Cat} \\ (\mathcal{F},\mathcal{I}) &\mapsto \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F}-,\mathcal{I}-),\mathfrak{M}) \\ \left((\mathcal{G},\mathcal{J}) \xrightarrow{(f,\alpha)} (\mathcal{F},\mathcal{I})\right) &\mapsto (f,\alpha)^* \end{aligned}$$

Lemma 4.4.44 quoted above reads as follows:

[Ayoub, 4.4.3. Lem.4.4.44 p.524]

- Given $\mathcal{C}$, a bicomplete category, $f : \mathcal{S} \to \mathcal{S}'$, a functor between small categories, we have

$$f_* : \mathbf{PreShv}(\mathcal{S}', \mathcal{C}) \to \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$$
$$H' \mapsto H' \circ f$$

- f_* admits a left adjoint

$$\begin{array}{rcl} f^* : \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) & \to & \mathbf{PreShv}(\mathcal{S}', \mathcal{C}) \\ K & \mapsto & \left(f^*K : \mathrm{Ob}(\mathcal{S}') \ni U' \quad \mapsto \quad \operatorname*{Colim}_{(U' \to f(U)) \in \mathrm{Ob}(U' \backslash \mathcal{S})} K(U) \in \mathrm{Ob}(\mathcal{C}) \right) \end{array}$$

Forgotten (?) to be stated in Ayoub, because these are commoon senses?

- It might be useful to regard f^*K as a left Kan extension of K along $f : \mathcal{S} \to \mathcal{S}'$.
- Then, similarly, we may construct a right adjoint

$$\begin{array}{rcl} f^! : \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) & \to & \mathbf{PreShv}(\mathcal{S}', \mathcal{C}) \\ K & \mapsto & f^!K \end{array}$$

where $f^!K$ is a right Kan extension of K along $f : \mathcal{S} \to \mathcal{S}'$.

- In summary, starting with the evident direct image functor of the categories of presheaes valued in a cocomplete category f_*, we may construct a composite of adjunctions via left and right Kan extensions:

$$\begin{cases} f^* \dashv f_* \dashv f^! \\ \mathbf{PreShv}(\mathcal{S}', \mathcal{C}) \begin{smallmatrix} \xleftarrow{f^*} \\ \xrightarrow{f_*} \\ \xleftarrow[f^!]{} \end{smallmatrix} \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \end{cases}$$

- If we have an adjunction:

$$f \dashv g, \quad \mathcal{S} \begin{smallmatrix} \xrightarrow{f} \\ \xleftarrow[g]{} \end{smallmatrix} \mathcal{S}'$$

then the above composite of adjunctions is expanded as follows:

$$\begin{cases} g^* \dashv f^* = g_* \dashv f_* = g^! \dashv f^! \\ \mathbf{PreShv}(\mathcal{S}', \mathcal{C}) \begin{smallmatrix} \xrightarrow{g^*} \\ \xleftarrow{f^* = g_*} \\ \xrightarrow{f_* = g^!} \\ \xleftarrow[f^!]{} \end{smallmatrix} \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \end{cases}$$

- If $(p : Y \to X) \in \mathbf{Ar}(\mathcal{S})$, then we get an adjunction:

$$\begin{array}{c} p : \mathcal{S}/X \leftrightarrows \mathcal{S}/Y : c_p \\ (U \to X) \mapsto (U \times_X Y \to Y) \\ (V \to Y \xrightarrow{p} X) \leftarrow\!\shortmid (V \to Y) \end{array}$$

For this, we get:

$$\begin{cases} (c_p)^* \dashv p^* = (c_p)_* \dashv p_* = (c_p)^! \dashv p^! \\ \mathbf{PreShv}(\mathcal{S}/Y, \mathcal{C}) \begin{smallmatrix} \xrightarrow{(c_p)^*} \\ \xleftarrow{p^* = (c_p)_*} \\ \xrightarrow{p_* = (c_p)^!} \\ \xleftarrow[p^!]{} \end{smallmatrix} \mathbf{PreShv}(\mathcal{S}/X, \mathcal{C}) \\ (p_*K)(U \to X) = K\,(p(U \to X)) = K\,(U \times_X Y \to Y) \\ (p^*H)(V \to Y) = ((c_p)_*H)\,(V \to Y) = H\,(c_p(V \to Y)) \\ \qquad = H(V \to Y \xrightarrow{p} X) = (Y \times_X H)(V \to Y) \end{cases}$$

From the last two equations, we get:

$$\boxed{p^*H = Y \times_X H, \text{ the pullback}; \qquad p_*K = K \circ p^*}$$

- More generlly, if $(\phi : \mathcal{G} \to \mathcal{F}) \in \mathbf{Ar}\,(\mathbf{PreShv}(\mathcal{S}, \mathcal{C}))$ with $\mathcal{G}, \mathcal{F} \in \mathrm{Ob}(\mathcal{S})$, then we can generalize the above adjunction to this more general case:

$$\begin{cases} \phi^* \dashv \phi_* \\ \mathbf{PreShv}(\mathcal{S}/\mathcal{G}, \mathcal{C}) \begin{smallmatrix} \xleftarrow{\phi^*} \\ \xrightarrow[\phi_*]{} \end{smallmatrix} \mathbf{PreShv}(\mathcal{S}/\mathcal{F}, \mathcal{C}) \\ \boxed{\phi^*H = \mathcal{G} \times_{\mathcal{F}} H, \text{ the pullback}; \qquad \phi_*K = K \circ \phi^*} \end{cases}$$

Observe the following two elementary, but important properties:

$$\begin{array}{c} (\phi_1 \circ \phi_2)^* = (\phi_2)^* \circ (\phi_1)^*; \ (\phi_1 \circ \phi_2)_* = (\phi_1)_* \circ (\phi_2)_* \\ \hom_{\mathcal{G}}(\phi^*H, K) = K(\phi^*H) = (\phi_*K)(H) = \hom_{\mathcal{F}}(H, \phi_*K) \end{array}$$

[Ayoub, Prop. 4.5.4, p.533]

Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schemes.

- The functor $(f,\alpha)^*$ admits a right adjoint $(f,\alpha)_*$:

$$(f,\alpha)^* : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathfrak{M}) \rightleftarrows \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{G},\mathcal{J}),\mathfrak{M}) : (f,\alpha)_*$$

- If (f,α) is smooth argument by argument, the functor $(f,\alpha)^*$ admits a left adjoint $(f,\alpha)_\sharp$:

$$(f,\alpha)_\sharp : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{G},\mathcal{J}),\mathfrak{M}) \rightleftarrows \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathfrak{M}) : (f,\alpha)^*$$

Proof. Since

$$\begin{aligned}(f,\alpha)^* := f^* \circ \alpha^* : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathfrak{M}) &\xrightarrow{\alpha^*} \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}),\mathfrak{M}) \\ &\xrightarrow{f^*} \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{G},\mathcal{J}),\mathfrak{M}),\end{aligned}$$

we shall construct the adjoints separately:

The case of $f^* : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}),\mathfrak{M}) \to \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{G},\mathcal{J}),\mathfrak{M})$,
Since the functor f^* is the inverse image following the functor

$$f = (-\times_{\mathcal{F}} \mathcal{G}) : \mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}),$$

- Its right adjoint f_* is the direct image functor:

$$\begin{aligned}f_* : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{G},\mathcal{J}),\mathfrak{M}) &\to \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J})) \\ H &\mapsto H\circ f\end{aligned}$$

- When f is smooth, the functor

$$\begin{aligned}f = -\times_{\mathcal{F}}\mathcal{G} : \mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}) &\to \mathrm{Sm}\,/(\mathcal{G},\mathcal{J}) \\ ((V\to\mathcal{F}(\alpha(j)))\,,j) &\mapsto \big((V\times_{\mathcal{F}(\alpha(j))}\mathcal{G}(j),j)\big),\end{aligned}$$

admits a right adjoint c_f :

$$\begin{aligned}f = -\times_{\mathcal{F}}\mathcal{G} : \mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}) &\rightleftarrows \mathrm{Sm}\,/(\mathcal{G},\mathcal{J}) : c_f \\ (V\to\mathcal{G}(j)\to\mathcal{F}(\alpha(j),j)) &\leftarrow\!\mid (V\to\mathcal{G}(j),j))\end{aligned}$$

Thus the left adjoint $f_\sharp$ of f^* is given by c_f^*, and furthermore, we hae $f^* = (c_f)_*$.

The case of $\alpha^* : \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F},\mathcal{I}),\mathfrak{M}) \to \mathbf{PreShv}\,(\mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}),\mathfrak{M})$
Since the functor α^* is the direct image following the functor

$$\begin{aligned}\overline{\alpha} : \mathrm{Sm}\,/(\mathcal{F}\circ\alpha,\mathcal{J}) &\to \mathrm{Sm}\,/(\mathcal{F},\mathcal{I}) \\ ((U\to\mathcal{F}(\alpha(j)))\,,j) &\mapsto ((U\to\mathcal{F}(\alpha(j)))\,,\alpha(j)),\end{aligned}$$

- Its left adjoint $\alpha_\sharp$ is given by $\overline{\alpha}^*$.
- Its right adjoitn α_* is gien by $\overline{\alpha}^!$.

□

[Ayoub, Prop.4.5.4, p.533; Lem.4.5.5, Lem.4.5.6, Lem.4.5.7, p.534]

- Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schemes.
  - The functor $(f,\alpha)^*$ admits a right adjoint $(f,\alpha)_*$.
  - If (f,α) is smooth argument by argument, the functor $(f,\alpha)^*$ admits a left adjoint $(f,\alpha)_\sharp$.

Proof. Use the decomposition $(f,\alpha) = \alpha \circ f$:

The case of $f : (\mathcal{G},\mathcal{J}) \to (\mathcal{F}\circ\alpha,\mathcal{J})$ f^* is the inverse image w.r.t. the functor

$$f = (- \times_{\mathcal{F}} \mathcal{G}) : \mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}) \to \mathrm{Sm}/(\mathcal{G},\mathcal{J})$$

It admits a right adjoint, i.e. the direct image functor which to a presheaf H on $\mathrm{Sm}/(\mathcal{G},\mathcal{J})$ associates $H \circ f$.

When the morphism $f(j)$ are smooth, the functor $f : \mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}) \to \mathrm{Sm}/(\mathcal{G},\mathcal{J})$ admits a right adjoint c_f which to $(V \to \mathcal{G}(j), j)$ associates the pair $(V \to \mathcal{G}(j) \to \mathcal{F}(\alpha(j),j))$. The left adjoint $f_\sharp$ of f^* is then given by c_f^*. Then furhtermore $f^* = (c_f)_*$.

The case of $\alpha : (\mathcal{F}\circ\alpha,\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ The functor α^* is constructed as the direct image of the functor $\overline{\alpha}$. It thus admits a left adjoint $\alpha_\sharp = \overline{\alpha}^*$ and the right adjoint $\alpha_* = \overline{\alpha}^!$. □

- Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-morphisms. For $i \in \mathrm{Ob}(\mathcal{I})$ we form the square boundary in $\mathbf{DiaSch}/S$:

$$\begin{tikzcd}
(\mathcal{G}/i,\mathcal{J}/i) \arrow[r, "(\mathrm{id},u_i)"] \arrow[d, "(f/i)"'] \arrow[dr, phantom, "\not\Downarrow_r"] & (\mathcal{G},\mathcal{J}) \arrow[d, "(f,\alpha)"] \\
\mathcal{F}(i) \arrow[r, "(\mathrm{id}_{\mathcal{F}(i)},i)"'] & (\mathcal{F},\mathcal{I})
\end{tikzcd}$$

Then the natural transformation

$$\left(\mathrm{id}_{\mathcal{F}(i)},i\right)^*(f,\alpha)_* \to (f/i)_*(\mathrm{id},u_i)^*$$

is invertible.

- When $\mathcal{F}$ and $\mathcal{G}$ are constant valued at the S-scheme F, we can form the square face:

$$\begin{tikzcd}
(F,i\backslash\mathcal{J}) \arrow[r, "u_i"] \arrow[d, "\alpha/i"'] \arrow[dr, phantom, "{}^r\!\!\not\Uparrow"] & (F,\mathcal{J}) \arrow[d, "\alpha"] \\
(F,i) \arrow[r, "i"] & (F,\mathcal{I})
\end{tikzcd}$$

Then the natural transformation

$$(\alpha/i)_\sharp u_i^* \to i^*\alpha_\sharp$$

is invertible.

- Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism of $\mathcal{I}$-diagrams of S-schemes. For a functor $\alpha : \mathcal{J} \to \mathcal{I}$, we form the commutatie square:

$$\begin{tikzcd}
(\mathcal{G}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"] \arrow[d, "f_{|\mathcal{J}}"'] & (\mathcal{G},\mathcal{I}) \arrow[d, "f"] \\
(\mathcal{F}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"'] & (\mathcal{F},\mathcal{I})
\end{tikzcd}$$

Then the natural transformation

$$\alpha^* f_* :\to \left(f_{|\mathcal{J}}\right)_* \alpha^*$$

is invertible. Suppose further f is cartesian and smooth level by level. Then the following is invertible:

$$\left(f_{|\mathcal{J}}\right)_\sharp \alpha^* \to \alpha^* f_\sharp$$

3.2 Results involving monoidal structures

[Ayoub, Def.2.1.79, p.210; Def.2.1.81, Rem.2.1.83, p.211]

- A monoidal category is a triple $(\mathcal{C}, \otimes, \sigma)$ with:
  - $\mathcal{C}$ a category,
  - $-\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a covariant functor,
  - $\sigma : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ a familty of natural isomorphisms on $(A, B, C) \in \mathrm{Ob}(\mathcal{C})^3$.

  The isomorphisms σ are caled the associativity isomorphisms . They satisfy certain axioms including the pentagon aximo saying that all the isomorphisms between the abjects $(A \otimes B) \otimes (C \otimes D)$ and $((A \otimes B) \otimes C) \otimes D$ built from the associativity isomorphisms (and their inverses) are equal.
- A symmetric monoidal category is a quadruplet $(\mathcal{C}, \otimes, \sigma, \tau)$ with:
  - $(\mathcal{C}, \otimes, \sigma)$ is a monoidal category,
  - $\tau : A \otimes B \xrightarrow{\sim} B \otimes A$ a faimily of the natural isomorphisms on $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$.

  The isomorphisms τ are callled the commutativity isomorphisms . They verify certain axiomos includin the equality $\tau \circ \tau = \mathrm{id}$. The supplementary axioms are imposed to describe the compatabilities between the associativity and the commutativity axioms.
- For a monoidal category $(\mathcal{C}, \otimes, \sigma)$, a unit object of $\mathcal{C}$ is a triple $(\mathbf{1}, u_g, u_d)$ with:
  - $\mathbf{1}$ is an object of $\mathcal{C}$,
  - $u_g : \mathbf{1} \otimes A \xrightarrow{\sim} A$ and $u_d : A \otimes \mathbf{1} \xrightarrow{\sim} A$ are natural isomorphisms on $A \in \mathrm{Ob}(^{C}cal)$.

  The isomorphisms u_g and u_d are called the left unit isomorphisms and the right unit isomorphisms respectively They must verify certain conditins. Note the two isomorphisms $u_g, u_d : \mathbf{1} \to \mathbf{1} \otimes \mathbf{1}$ coincide.
- For a symmetric monoidal category $(\mathcal{C}, \otimes, \sigma, \tau)$, a unit object of $\mathcal{C}$ is a unit object $(\mathbf{1}, u_g, u_d)$ of the monoidal category $(\mathcal{C}, \otimes, \sigma)$ verifying certain compatibilities supplementary with th commutation isomrophisms. Note for example that the following diagram:

  $$\begin{tikzcd} \mathbf{1} \otimes A \arrow[rr, "\tau"] \arrow[dr, "u_g"'] & & A \otimes \mathbf{1} \arrow[dl, "u_d"] \\ & A & \end{tikzcd}$$

  is commutative. Thus, the isomorphisms u_g and u_d are deduced from each other.
- A monoidal (resp. symmetric monoidal) category provided with a unit object is called a unital monoidal (resp. unital symmetric monoidal) category .
- For a monoidal category $(\mathcal{C}, \otimes, \sigma)$, the functor $\otimes$ induces a covariant functor:

  $$\otimes^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{O}^{\mathrm{op}} = (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$$

  and we take for the corresponding associativity isomorphisms the arrows $(\sigma^{-1})^{\mathrm{op}}$:

  $$\left(\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "(-\otimes-)\otimes-"] \arrow[r, bend right, "-\otimes(-\otimes-)"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathcal{C} \end{tikzcd}\right) \Longleftrightarrow \left(\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} & \mathcal{C} \arrow[l, bend right, "(-\otimes-)\otimes-"'] \arrow[l, bend left, "-\otimes(-\otimes-)"] \arrow[l, phantom, "\Uparrow\sigma^{-1}"] \end{tikzcd}\right) \Longleftrightarrow \left(\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "(-\otimes^{\mathrm{op}}-)\otimes^{\mathrm{op}}-"] \arrow[r, bend right, "-\otimes^{\mathrm{op}}(-\otimes^{\mathrm{op}}-)"'] \arrow[r, phantom, "\Downarrow(\sigma^{-1})^{\mathrm{op}}"] & \mathcal{C}^{\mathrm{op}} \end{tikzcd}\right)$$

  A new monoidal category $(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, (\sigma^{-1})^{\mathrm{op}})$, which we call the opposite monoidal category of $(\mathcal{C}, \otimes, \sigma)$.
- On the other hand, using

  $$perm : \mathcal{C} \times \mathcal{C} \to \mathcal{C}; \quad (A, B) \mapsto (B, A)$$

  we define a functor $\otimes^\circ$:

  $$\begin{cases} \otimes^\circ = \otimes \circ per : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \\ A \otimes^\circ B = B \otimes A \end{cases}$$

  and a natural isotransformation σ°:

  $$\left(\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "(-\otimes^\circ-)\otimes^\circ-"] \arrow[r, bend right, "-\otimes^\circ(-\otimes^\circ-)"'] \arrow[r, phantom, "\Downarrow\sigma^\circ"] & \mathcal{C} \end{tikzcd}\right) := \left(\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "{(A,B,C)\mapsto(C,B,A)}"] & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "-\otimes(-\otimes-)"] \arrow[r, bend right, "(-\otimes-)\otimes-"'] \arrow[r, phantom, "\Downarrow\sigma^{-1}"] & \mathcal{C} \end{tikzcd}\right)$$

  We get a new monoidal category $(\mathcal{C}, \otimes^\circ, \sigma^\circ)$, which we call the $\otimes$-opposite monoidal category of $(\mathcal{C}, \otimes, \sigma)$.
- When $(\mathcal{C}, \otimes, \sigma, \tau)$ is a symmetric monoidal category, the quadruplets

  $$(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, (\sigma^{-1})^{\mathrm{op}}, \tau^{\mathrm{op}}) \quad \text{and} \quad (\mathcal{C}, \otimes^\circ, \sigma^\circ, \tau^\circ := \tau^{-1})$$

  are symmetric monoidal categories, which we respectively call the opposite (resp. $\otimes$-opposite) symmetric monoidal category of $(\mathcal{C}, \otimes, \sigma, \tau)$.
- A unit object $(\mathbf{1}, u_g, u_d)$ of $(\mathcal{C}, \otimes, \sigma)$ (resp. $(\mathcal{C}, \otimes, \sigma, \tau)$) induces the unit objects $(\mathbf{1}, (u_g^{-1})^{\mathrm{op}}, (u_d^{-1})^{\mathrm{op}})$ and $(\mathbf{1}, u_d, u_g)$ of the monoidal categories $(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, (\sigma^{-1})^{\mathrm{op}})$ and $(\mathcal{C}, \otimes^\circ, \sigma^\circ)$. (resp. symmetric monoidal categories $(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, (\sigma^{-1})^{\mathrm{op}}, \tau^{\mathrm{op}})$ and $(\mathcal{C}, \otimes^\circ, \sigma^\circ, \tau^\circ := \tau^{-1})$).
- We can view a monoidal category as a 2-category (not necessarily strict or unital) having a sole object. In this way, the opposite and $\otimes$-opposite monoidal categories correspond to the 2-opposite and the 1-opposite 2-categories, respectively.

(p$\mathfrak{Mono}$), ($\mathfrak{Mono}$), (pc$\mathfrak{Mono}$), (c$\mathfrak{Mono}$) [Ayoub, Def.2.1.85, p.212; Def.2.1.86, Def.2.1.87, p.213; p.214]

- Let $(\mathcal{C}, \otimes, \sigma)$ and $(\mathcal{C}', \otimes', \sigma')$ be two monoidal categories. peudo-monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (f, a) of the form:
  - a functor $f : \mathcal{C} \to \mathcal{C}'$,
  - morphisms $a : f(A) \otimes' f(B) \to f(A \otimes B)$ natural in $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$, such that the following diagram:

$$\begin{tikzcd} (f(A) \otimes' f(B)) \otimes' f(C) \arrow[r] \arrow[d] & f(A \otimes B) \otimes' f(C) \arrow[r] & f((A \otimes B) \otimes C) \arrow[d] \\ f(A) \otimes' (f(B) \otimes' f(C)) \arrow[r] & f(A) \otimes' f(B \otimes C) \arrow[r] & f(A \otimes (B \otimes C)) \end{tikzcd}$$

  is commutative for any $(A, B, C) \in \mathrm{Ob}(\mathcal{C})^3$.

  The morphisms a are sometimes called the coupling morphism of f. When they are invertible, we say that (f, a) is monoidal .

  A natural transformation between two pseudo-monoidal functors (f_1, a_1) and (f_2, a_2) is a natural transformation $f_1 \dashv f_2$ s.t. the following diagram commutes for any $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$:

$$\begin{tikzcd} f_1(A) \otimes f_1(B) \arrow[r] \arrow[d] & f_2(A) \otimes f_2(B) \arrow[d] \\ f_1(A \otimes B) \arrow[r] & f_2(A \otimes B) \end{tikzcd}$$

- Keeping the above notations and suppose given the unit objects $(\mathbf{1}, u_g, u_d)$ and $(\mathbf{1}', u'_g, u'_d)$ of $\mathcal{C}$ and $\mathcal{C}'$ respectively. A pseudo-unital pseudo-monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a triplet (f, a, e) such that:
  - (f, a) is a pseudo-monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$,
  - $e : \mathbf{1}' \to f(\mathbf{1})$ is an arrow compatible with the left and right unit isomorphisms, i.e. such that the following diagram is commutative:

$$\begin{tikzcd} \mathbf{1}' \otimes' f(A) \arrow[r, "e"] \arrow[d, "u'_g", "\sim"'] & f(\mathbf{1}) \otimes' f(A) \arrow[r] & f(\mathbf{1} \otimes A) \arrow[d, "\sim"', "u_g"] \\ f(A) \arrow[rr, equal] & & f(A) \end{tikzcd}$$

  as well as its analogue for u_d and u'_d.

  When the morphisms a, e are invertible, (f, a, e) is called a unital monoidal functor . A natural transformatin between two pseudo-unital pseudo-monoidal functors (f_1, a_1, e_1) and (f_2, a, e_2) is a natural transformation between pseudo-monoidal functors (f_1, a_1) and (f_2, a_2) s.t. moreover the following square is commutative:

$$\begin{tikzcd} \mathbf{1}' \arrow[r, "e_1"] \arrow[d, equal] & f_1(\mathbf{1}) \arrow[d] \\ \mathbf{1}' \arrow[r, "e'_2"] & f_2(\mathbf{1}) \end{tikzcd}$$

- Let $(\mathcal{C}, \otimes, \sigma, \tau)$ and $(\mathcal{C}', \otimes', \sigma', \tau')$ be two symmetric monoidal (resp. unital symmetric monoidal) categories. A symmetric pseudo-monoidal (resp. pseudo-unital and pseudo-monoidal) functor from $\mathcal{C}$ to $\mathcal{C}'$ is a pseudo-monoidal (resp. pserudo-unital pseudo-unital) functor (f, a) such that the following diagram:

$$\begin{tikzcd} f(A) \otimes' f(B) \arrow[r, "a"] \arrow[d, "\tau", "\sim"'] & f(A \otimes B) \arrow[d, "\sim"', "\tau"] \\ f(B) \otimes' f(A) \arrow[r] & f(B \otimes A) \end{tikzcd}$$

  is commutative for any $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$.

  A natural transformation between two symmetric pseudo-monoidal (resp. symmetric pseudo-unital pseudo-monoidal) functors is simply a natural transformation between the underlying pseudo-monoidal (resp. pseudo-unital pseudo-monoidal) functors.

- Get the notion of pseudo-comonoidal functor from that of pseudo-monoidal functor by passing to the oppsote categories; if $(\mathcal{C}, \otimes, \sigma)$ and $(\mathcal{C}', \otimes', \sigma')$ are two monoidal categories, a pseudo-comonoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (f, a):
  - a functor $f : \mathcal{C} \to \mathcal{C}'$,
  - morphisms $a : f(A \otimes B) \to f(A) \otimes' f(B)$ natural in $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$, compatible in the evident manner with the associativity isomorphisms.

  The morphisms a are sometimes called cocouplement morphisms of f. When they are invertible, we say that (f, a) is comonoidal .

  A natural transformation between two pseudo-comonoidal functors (f_1, a_1) and (f_2, a_2) is a natural transformation $f_1 \dashv f_2$ s.t. the following diagram commutates for any $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$:

$$\begin{tikzcd} f_1(A \otimes B) \arrow[r] \arrow[d] & f_2(A \otimes B) \arrow[d] \\ f_1(A) \otimes f_1(B) \arrow[r] & f_2(A) \otimes f_2(B) \end{tikzcd}$$

- Get the notion of pseudo-counital pseudo-comonoidal functor (resp. counital comonoidal functor) by passing to opposite categories. Similary, get the notion of symmetric pseudo-comonoidal functor between symmetric monoidal categories.

- The monoidal categories with pseudo-monoidal (resp. pseudo-comonoidal) functors and their natural transformations for ma strict 2 category: (p$\mathfrak{Mono}$) (resp. (pc$\mathfrak{Mono}$)). Similarly, get the sub-2-category ($\mathfrak{Mono}$) (resp. (c$\mathfrak{Mono}$)) where we take only the monoidal (resp. comonoidal) functors. The association $(f, a) \rightsquigarrow (f, a^{-1})$ defines an isomorphism between ($\mathfrak{Mono}$) and (c$\mathfrak{Mono}$).

$\mathfrak{Mod}_g \to \mathbf{p}\mathfrak{Mono}$, $\mathfrak{Mod}_d, \mathbf{c}\mathfrak{Mod}_d$. [Ayoub, Def.2.1.93, p.216; Def.2.1.94, p.217]

Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be two monoidal (resp. unital monoidal with unit objects $\mathbf{1}$ and $\mathbf{1}'$ respectively) categories.

1: Let (f, a) (resp. (f, a, e)) be a pseudo-monoidal (resp. pseudo-unital and pseudo-monoidal) functor. We call by a <u>left f-module (resp. unital f-module)</u> a pair (l, b) with:

- $l : \mathcal{C} \to \mathcal{C}'$ a functor,
- $b : f(A) \otimes' l(B) \to l(A \otimes B)$ natural morphisms in $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$,

s.t. for any $(A, B, C) \in \mathrm{Ob}(\mathcal{C})^3$ the following diagram:

$$\begin{tikzcd} (f(A) \otimes' f(B)) \otimes' l(C) \arrow[r, "a"] \arrow[d, "\sigma'"'] & f(A \otimes B) \otimes' l(C) \arrow[r, "b"] & l((A \otimes B) \otimes C) \arrow[d, "\sigma"] \\ f(A) \otimes' (f(B) \otimes l(C)) \arrow[r, "b"] & f(A) \otimes' l(B \otimes C) \arrow[r, "b"] & l(A \otimes (B \otimes C)) \end{tikzcd}$$

is commutative (resp. and such that the following composite is equal to the identity of the functor l):

$$l(-) \xrightarrow{u_g'^{-1}} \mathbf{1}' \otimes l(-) \to f(\mathbf{1}) \otimes l(-) \xrightarrow{b} l(\mathbf{1} \otimes -) \xrightarrow{u_g} l(-)$$

We have the notion of <u>$\otimes$-dual of right f-module (resp. unital f-module)</u> . A <u>morphism of f-modules</u> from (l, b) to (l', b') is a natural transformation from l to l' compatible in the evident sense with the morphisms b and b'.

2: Let (g, a) (resp. (g, a, e)) be a pseudo-comonoidal (resp. pseudo-counital pseudo-comonoidal) functor. We call by <u>left g-comodule</u> a pair (k, c) with:

- $k : \mathcal{C} \to \mathcal{C}'$ a functor,
- $c : k(A \otimes B) \to g(A) \otimes' k(B)$ natural morphisms in $(A, B) \in \mathrm{Ob}(\mathcal{C})^2$,

such that the dual conditions of **1** are verified.

We also have the notion of <u>right g-comodule (resp. counital g-comodule)</u> . A <u>morphism of g-comodules</u> from (k, c) to (k', c') is a natural transformation from k to k' compatible in the evident sense with the morphisms c, c'.

1': A <u>left module from $\mathcal{C}$ to $\mathcal{C}'$</u> $[f, l]$ is a quadruplet (f, l, a, b) with (f, a) a pseudo-monoidal functor and (l, b) a left f-module. The <u>category of left module from $\mathcal{C}$ to $\mathcal{C}'$</u> is denoted by $\mathbf{Mod}_g(\mathcal{C}, \mathcal{C}')$. An arrow of $\mathbf{Mod}_g(\mathcal{C}, \mathcal{C}')$ is a pair $(u, v) : (f_1, l_1, a_1, b_1) \to (f_2, l_2, a_2, b_2)$ with $u : f_1 \to f_2$ a monoidal natural transformation and $v : l_1 \to l_2$ a natural transformation such that the following diagram is commutative for any $(a, B) \in \mathrm{Ob}(\mathcal{C})^2$:

$$\begin{tikzcd} f_1(A) \otimes' l_1(B) \arrow[r] \arrow[d, "u \otimes' v"'] & l_1(A \otimes B) \arrow[d, "v"] \\ f_2(A) \otimes' l_2(B) \arrow[r] & l_2(A \otimes B) \end{tikzcd}$$

Being given a third monoidal category $(\mathcal{C}'', \otimes'')$, there exists a functor of composition:

$$\mathbf{Mod}_b(\mathcal{C}', \mathcal{C}'') \times \mathbf{Mod}_g(\mathcal{C}, \mathcal{C}') \to \mathbf{Mod}_g(\mathcal{C}, \mathcal{C}'')$$
$$([f', l'] = (f', l', a', b'), [f, l] = (f, l, a, b)) \mapsto [f' \circ f, l' \circ l] = (f' \circ f, l' \circ l, a'', b'')$$

with $(f' \circ f, a'')$ the pseudo-monoidal functor composite of f and f' (see Rem.2.1.89) and b'' the composite:

$$f' \circ f(A) \otimes l' \circ l(B) \to l'\,(f(A) \circ l(A)) \to l' \circ l(A).$$

We thus obtain the 2-category $(\mathfrak{Mod}_g)$ whose objects are monoidal categories and whose 1-morphisms are left modules. We thave an evident forgetfull 1-covariant 2-covariant 2-functor:

$$\mathfrak{Mod}_g \to \mathbf{p}\mathfrak{Mono}$$
$$(f, l, a, b) \mapsto (f, a).$$

2': We have the dual notion of left comodules from $\mathcal{C}$ to $\mathcal{C}'$. We also form the strict 2-category of comodules $\mathbf{c}\mathfrak{Mod}_g$. The notions of$\otimes$-duals, right modules, and right comodules also organize two strict 2-categories $\mathfrak{Mod}_d$ and $\mathbf{c}\mathfrak{Mod}_d$.

$\mathfrak{Mod}, \mathfrak{cMod}$ [Ayoub, Def.2.1.95, p.217]

1: Keeing he notations of Definition 2.1.93, a f-bimodule is a triplet (l, b_g, b_d) such that (l, b_g) is a left f-module and (l, b_d) is a right f-module and such that the following diagram is commutative:

$$\begin{tikzcd}
(f(A) \otimes' l(B)) \otimes' f(C) \arrow[r, "b_g"] \arrow[d, "\sim"'] & l(A \otimes B) \otimes' f(C) \arrow[r, "b_d"] & l((A \otimes B) \otimes C) \arrow[d, "sim"] \\
f(A) \otimes' (l(B) \otimes' f(C)) \arrow[r, "b_d"] & f(A) \otimes' l(B \otimes C) \arrow[r, "b_g"] & l(A \otimes (B \times C))
\end{tikzcd}$$

for any $(A, B, C) \in \mathrm{Ob}(\mathcal{C})^3$. We also define the notion of bimodule from $\mathcal{C}$ to $\mathcal{C}'$ as well as the 2-category of bimodules $\mathfrak{Mod}$.

2: We also have the dual notion of g-bicomodules and of bicomodules from $\mathcal{C}$ to $\mathcal{C}'$. We obtain $\mathfrak{cMod}$ the 2-category of bicomodules.

[Ayoub, Prop.2.1.97, p.218; Prop.2.1.98, p.219]

- Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be two monoidal categories and $(f, \alpha) : \mathcal{C} \to \mathcal{C}'$ be a pseudo-monoidal functor.

1: Suppose given a left f-module (l, b) and a left adjoint k of the functor l. We define a morphism $c : k\,(f(A) \otimes' B') \to A \otimes k(B')$, natural on $A \in \mathrm{Ob}(\mathcal{C})$ and $B' \in \mathrm{Ob}(\mathcal{C}')$, by the composite:

$$k\,(f(A) \otimes' B') \to k\,(f(A) \otimes' lk(B')) \xrightarrow{b} kl\,(A \otimes k(B')) \to A \otimes k(B')$$

Then the following diagram:

$$\begin{tikzcd}
k\,(f(A) \otimes' (f(B) \otimes' C')) \arrow[r] \arrow[d, "\sim"'] & A \otimes k\,(f(B) \otimes' C') \arrow[r] & A \otimes (B \otimes k(C')) \arrow[d, "\sim"] \\
k\,((f(A) \otimes' f(B)) \otimes' C') \arrow[r] & k\,(f(A \otimes B) \otimes' C') \arrow[r] & (A \otimes B) \otimes k(C').
\end{tikzcd}$$

is commutative for any $(A, B, C') \in \mathrm{Ob}(\mathcal{C})^2 \times \mathrm{Ob}(\mathcal{C}')$.

2: Suppose given a f-bimodule (l, b_g, b_d) as well as a left adjoint k of l. Then the left (l, b_g) module and the right (l, b_d) module by **1**-the two transformations:

$$c_g : k\,(f(A) \otimes' B') \to A \otimes k(B') \text{ and } c_d : k\,(A' \otimes' f(B)) \to k(A') \otimes B$$

which are natural on $(A, B, A', B') \in \mathrm{Ob}(\mathcal{C})^2 \times (\mathcal{C}')^2$. The following diagram:

$$\begin{tikzcd}
k\,(f(A) \otimes' (B' \otimes' f(C))) \arrow[r, "c_g"] \arrow[d, "\sim"'] & A \otimes k\,(B' \otimes' f(C)) \arrow[r, "c_d"] & A \otimes (k(B') \otimes C) \arrow[d, "\sim"] \\
k\,((f(A) \otimes' B') \otimes' f(C)) \arrow[r, "c_d"] & k\,(f(A) \otimes' B') \otimes C \arrow[r, "c_g"] & (A \otimes k(B')) \otimes C
\end{tikzcd}$$

is natural fora an $(A, B', C) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}') \times \mathrm{Ob}(\mathcal{O})$.

- Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be two monoidal categories and $(g, \alpha) : \mathcal{C} \to \mathcal{C}'$ be a pseudo-comonoidal functor.

1: Suppose given a left g-comodule (k, b) and a right adjoint l of the functor k. We define a morphism $c : A \otimes l(B') \to l\,(g(A) \otimes' B')$, natural on $A \in \mathrm{Ob}(\mathcal{C})$ and $B' \in \mathrm{Ob}(\mathcal{C}')$, by the composite:

$$A \otimes l(B') \to lk\,(A \otimes l(B')) \xrightarrow{b} l\,(g(A) \otimes' kl(B')) \to l\,(g(A) \otimes' B')$$

Then the following diagram:

$$\begin{tikzcd}
A \otimes (B \otimes l(C')) \arrow[r] \arrow[d, "\sim"'] & A \otimes l\,(g(B) \otimes' C') \arrow[r] & l\,(g(A) \otimes' (g(B) \otimes' C')) \arrow[d, "\sim"] \\
(A \otimes B) \otimes l(C') \arrow[r] & l\,(g(A \otimes B) \otimes' C') \arrow[r] & l\,((g(A) \otimes' g(B)) \otimes' C')
\end{tikzcd}$$

is commutative for any $(A, B, C') \in \mathrm{Ob}(\mathcal{C})^2 \times \mathrm{Ob}(\mathcal{C}')$.

2: Suppose given a g-bicomodule (k, b_g, b_d) as well as a left adjoint l of k. Then the left (l, b_g) comodule and the right (l, b_d) comodule by **1**-the two transformations:

$$c_g : A \otimes l(B') \to l\,(g(A) \otimes' B') \text{ and } c_d : l(A') \otimes B \to l\,(A' \otimes' g(B))$$

which are natural on $(A, B, A', B') \in \mathrm{Ob}(\mathcal{C})^2 \times (\mathcal{C}')^2$. The following diagram:

$$\begin{tikzcd}
A \otimes (l(B') \otimes C) \arrow[r, "c_d"] \arrow[d, "\sim"'] & A \otimes l\,(B' \otimes' g(C)) \arrow[r, "c_g"] & l\,(g(A) \otimes' (B' \otimes' g(C))) \arrow[d, "\sim"] \\
(A \otimes l(B')) \otimes C \arrow[r, "c_g"] & l\,(g(A) \otimes' B') \otimes C \arrow[r, "c_d"] & l\,((g(A) \otimes' B') \otimes' g(C))
\end{tikzcd}$$

is natural for any $(A, B', C) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}') \times \mathrm{Ob}(\mathcal{O})$.

$\mathfrak{Proj}_g \to \mathfrak{pMono}, \mathfrak{Proj}_d, \mathfrak{cProj}_d, \mathfrak{Proj}, \mathfrak{cProj}$. [Ayoub, Def.2.1.99, Def.2.1.100, p.220; Def.2.1.101, p.221]

Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be two monoidal categories.

1: Let (f, a) be a pseudo-monoidal functor. We call by <u>left f-projector</u> a pair (k, c) with:

- $k : \mathcal{C}' \to \mathcal{C}$ a functor,
- $c : k\,(f(A) \otimes' B') \to A \otimes k(B')$ natural morphisms in $A \in \mathrm{Ob}(\mathcal{C})$ and $B' \in \mathrm{Ob}(\mathcal{C}')$,

s.t. the following diagram commutes for any $(A, B, C') \in \mathrm{Ob}(\mathcal{C})^2 \times \mathrm{Ob}(\mathcal{C}')$:

$$\begin{tikzcd}
k\,(f(A) \otimes' (f(B) \otimes' C')) \arrow[r] \arrow[d, "\sim"'] & A \otimes k\,(f(B) \otimes' C') \arrow[r] & A \otimes (B \otimes k(C')) \arrow[d, "\sim"] \\
k\,((f(A) \otimes' f(B)) \otimes' C') \arrow[r] & k\,(f(A \otimes B) \otimes' C') \arrow[r] & (A \otimes B) \otimes k(C')
\end{tikzcd}$$

We have the $\otimes$-dual notion of <u>right f-projector</u> .

A <u>morphism of f-projector from (k, c) to (k', c')</u> is a natural transformation from k to k' compatible (evidently) with the morphisms b, b'.

2: Let (g, a) be a pseudo-comonoidal functor. We call by <u>left g-coprojector</u> a pair (n, d) with:

- $n : \mathcal{C}' \to \mathcal{C}$ a functor,
- $d : A \otimes n(B') \to n\,(g(A) \otimes' B')$ natural morphisms in $A \in \mathrm{Ob}(\mathcal{C})$ and $B' \in \mathrm{Ob}(\mathcal{C}')$,

such that the dual conditions of **1** are verified.

We also have the $\otimes$-dual notion of <u>right g-coprojector</u> .

A <u>morphism of g-coprojector from (n, d) to (n', d')</u> is a natural transformation from n to n' compatible in the evident sense with the morphisms c and c'.

1': A <u>left projector $[f, k]$ from $\mathcal{C}'$ to $\mathcal{C}$</u> is a quadruplet (f, k, a, c) with $(f, a) : \mathcal{C} \to \mathcal{C}'$ a pseudo-monoidal and (k, c) a left f-projector. The category of left projectors from $\mathcal{C}'$ to $\mathcal{C}$ is denoted by $\mathsf{Proj}_g(\mathcal{C}', \mathcal{C})$. An arrow of $\mathsf{Proj}_g(\mathcal{C}', \mathcal{C})$ is a pair $(u, v) : (f_1, k_1, a_1, c_1) \to (f_2, k_2, a_2, c_2)$ with $u : f_2 \to f_1$ a monoidal natural transformation and $v : k_1 \to k_2$ a natural transformation s.t. the following diagram is commutative for any $(A, B') \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}')$:

$$\begin{tikzcd}
k_1\,(f_2(A) \otimes' B') \arrow[r, "v"] \arrow[d, equal] & k_2\,(f_2(A) \otimes' B') \arrow[r, "c_2"] & A \otimes k_2(B') \\
k_1\,(f_2(A) \otimes' B') \arrow[r, "u"] & k_1\,(f_1(A) \otimes' B') \arrow[r, "c_1"] & A \otimes k_1(B') \arrow[u, "v"']
\end{tikzcd}$$

Given a third monoidal category $(\mathcal{C}'', \otimes'')$, we get a composition functor:

$$\mathsf{Proj}_g(\mathcal{C}', \mathcal{C}) \times \mathsf{Proj}_g(\mathcal{C}'', \mathcal{C}') \to \mathsf{Proj}_g(\mathcal{C}'', \mathcal{C})$$
$$([f', k'] = (f', k', a', c'), [f, k] = (f, k, a, c)) \mapsto (f' \circ f, k \circ k', a'', c'')$$

with $(f' \circ f, a'')$ the composite monoidal functor (see Remark 2.1.89) and e'' the composite:

$$kk'\,(f'f(A) \otimes'' B'') \xrightarrow{c'} k\,(f(A) \otimes' k((B''))$$

We obtain the 2-category $(\mathfrak{Proj}_g)$ whose objects are monoidal categories and whose 1-morphisms are left projectors. Note the evident 1-contravariant 2-contravariant 2-functor:

$$\mathfrak{Proj}_g \to \mathfrak{pMono}; \quad [f, k] \mapsto f$$

2': Also get the dual notion of <u>left coprojector</u> , organising a 2-category $\mathfrak{cProj}_g$, and strict 2-categories $\mathfrak{Proj}_d$ and $\mathfrak{cProj}_d$, furnished by as $\otimes$-dual notions.

1": Aa <u>f-biprojector</u> is a triplet (k, c_g, c_d) such that (k, c_g) is a left f-projector and (k, c_d) a right f-projector and s.t. the following diagram is commutative for any $(A, B', C) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}') \times \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd}
k\,(f(A) \otimes' (B' \otimes' f(C))) \arrow[r, "c_g"] \arrow[d, "\sim"'] & A \otimes k\,(B' \otimes' f(C)) \arrow[r, "c_d"] & A \otimes (k(B') \otimes C) \arrow[d, "\sim"] \\
k\,((f(A) \otimes' B') \otimes' f(C)) \arrow[r, "c_d"] & k\,((f(A) \otimes' B') \otimes C \arrow[r, "c_g"] & (A \otimes k(B')) \otimes C
\end{tikzcd}$$

2": We also have the dual notion of <u>g-bicoprojectors</u> . We also define the notion of <u>biprojector (resp. bicoprojector) from $\mathcal{C}$ to $\mathcal{C}'$</u> . We denote by $\mathfrak{Proj}$ and $\mathfrak{cProj}$ the 2-categories of projectors and coprojectors.

[Ayoub, Def.2.1.119, p.230; Def.2.1.148, p.239; Def.2.1.149, p.240]

- Let $(\mathcal{C}, \otimes)$ be a monoidal category. We say it is left closed if for any object A of $\mathcal{C}$, the functor $A \otimes -$ admits a right adjoint. We say that $\mathcal{C}$ is right closed if for any object A of $\mathcal{C}$, the functor $- \otimes A$ admits a right adjoint.
- A monoidal category $\mathcal{C}$ is right closed if the $\otimes$-opposed category $\mathcal{C}^0$ is left closed and vice versa. So just to study one type of the closed monoidal categories.

  Thereafter, we consider mainly the right closed monnoidal categories. We denote by $\underline{\mathrm{Hom}}(A, -)$ the right adjoint of $- \otimes A$. There is thus isomorphisms:

$$\hom_{\mathcal{C}}(U \otimes A, V) \xrightarrow{\sim} \hom_{\mathcal{C}}(U, \underline{\mathrm{Hom}}(A, V))$$

  as well as the arrows:

$$ev : \underline{\mathrm{Hom}}(A, v) \otimes A \to V \quad \text{and} \quad \delta : U \to \underline{\mathrm{Hom}}(A, U \otimes A)$$

  natural in U and V of $\mathcal{C}$.
- When we will need to cisider right and left closed monoidal categories, we denote, to distinguish, $\underline{\mathrm{Hom}}_g(A, -)$ and $\underline{\mathrm{Hom}}_d(A, -)$ the respective right adjoints of $A \otimes -$ and $- \otimes A$.
- A monoidal (resp. symmetric monoidal) triangulated category is is an additive monoidal category $(\mathcal{T}, \otimes, \sigma)$ (resp. $(\mathcal{T}, \otimes, \sigma, \tau)$), with a structure of triangulated category on $\mathcal{T}$ as well as the isomorphisms:

$$A[+1] \otimes B \underset{\sim}{\xrightarrow{s_g}} (A \otimes B)[+1] \underset{\sim}{\xleftarrow{s_d}} A \otimes B[+1]$$

  which are natural on $(A, B) \in \mathrm{Ob}(\mathcal{T})^2$ and commute in the evident manner with the associativity (resp. the associativity and the commutativity) isomorphisms. Also, two supplementary axioms are imposed:

  - For any distinguished triangle $A \to B \to C \to A[+1]$ and any object D of $\mathcal{T}$ the two diagrams below:

$$A \otimes D \to B \otimes D \to C \otimes D \to (A \otimes D)[+1]$$
$$D \otimes A \to D \otimes B \to D \otimes C \to (D \otimes A)[+1]$$

    are distinguished. In other words, the functors $- \otimes D$ and $D \otimes -$ provided with the isomorphisms s_g and s_d respectively, are triangulated functors.
  - For any A and B of $\mathcal{T}$, the square below is commuative up to the multiplication by -1:

$$\begin{tikzcd}
A[+1] \otimes B[+1] \arrow[r] \arrow[d] & (A[+1] \otimes B)\,[+1] \arrow[d] \\
(A \otimes B[+1])\,[+1] \arrow[r] & (A \times B)[+2]
\end{tikzcd}$$

- Let $(\mathcal{T}, \otimes)$ and $(\mathcal{T}', \otimes')$ be two triangulated monoidal (resp. symmetric monoidal) categories. A pseudo-monoidal (resp. symmetric pseudo-monoidal) triangulated functor from $\mathcal{T}$ to $\mathcal{T}'$ is a pseudo-monoidal (resp. symmetric pseudo-monoidal) functor between underlying additive monoidal categories, which is triangulated and compatible with the isomorphisms s_g and s_d.

  Suppose unit objects are given in $\mathcal{T}$ and $\mathcal{T}'$. A triangulated pseudo-monoidal and pseudo-unital functor is simply a triangulated pseudo-monoidal functor provided with an arrow e which makes it also a pseudo-monoidal and pseudo-unital.
- A monoidal (resp. symmetric monoidal) triangulated derivateur is a triangulated derivateur $\mathbb{D}$ provided with the following supplementary data:
  - For each $I \in \mathrm{Ob}(\mathbf{Dia})$ a monoidal (resp. symmetric monoidal) category structure $(\mathbb{D}(I), \otimes_I, \sigma)$.
  - For each functor $u : A \to B$ of $\mathbf{Dia}$ a monoidal (resp. symmetric monoidal) functor structure on u^*.
- A triangulated unital monoidal (resp. symmetric monoidal) derivateur is a triangulated monoidal (resp. symmetric monoidal) derivateur provided with a unit object $\mathbf{1}_I \in \mathrm{Ob}(\mathbb{D}(I))$ for each $I \in \mathrm{Ob}(\mathbf{Dia})$ and an isomorphism $u^*\mathbf{1}_I \simeq \mathbf{1}_J$ for each $u : J \to I \in \mathsf{Fl}(\mathbf{Dia})$ making u^* a unital monoidal (resp. symmetric monoidal) functor.

[Ayoub, p.274; Def.2.3.1, p.274]

- $\mathfrak{MonoTR}$ (resp. $\mathfrak{uMonoTR}$) denotes the 2-category of the triangulated monoidal (resp. unital monoidal) categories, such that
  - 1-morphisms are monoidal (resp. unital monoidal) triangulated functors which commute with the isomorphisms s_g and s_d in Definition 2.1.148.
  - 2-functors, i.e. the natural transformations, are natural transformations of the monoidal (resp. unital monoidal) functors which are also natural transformations of triangulated functors.
- A monoidal (resp. unital monoidal) triangulated 2-functor is a 2-functor

$$(\mathsf{H}, \otimes) : \mathbf{Sch}/S \to \mathfrak{MonoTR}$$
$$(\text{resp. } \mathsf{H}, \otimes, \mathbf{1}) : \mathbf{Sch}/S \to \mathfrak{uMonoTR}$$

  which
  - to a quasi-projective S-scheme X associate a monoidal (respl. unital monoidal) triangulated category $(\mathsf{H}(X), \otimes_X)$ (resp. $(\mathsf{H}(X), \otimes_X, \mathbf{1}_X)$);
  - to a S-morphism $f : X \to Y$ asociate a monoidal (resp. unital monoidal) triangulated functor f^*.
- A monoidal triangulated 2-functor $(\mathsf{H}, \otimes)$ is called a stable homotopy monoidal 2-functor if the following two conditions are satisfied:
  - when composed to the right by the forgetful (strict) 2-functor: $\mathfrak{MonoTR} \to \mathfrak{TR}$ (resp. $\mathfrak{uMonoTR} \to \mathfrak{TR}$) we obtain a stable homotopy 2-functor.
  - (projection formula) Let $f : Y \to X$ be a smooth S-morphism. The two morphisms (see Proposition 2.1.97) :

$$p_g : f_\sharp\left(f^*(A) \otimes_Y B'\right) \to A \otimes_X f_\sharp(B')$$
$$p_d : f_\sharp\left(A' \otimes_Y f^*(B)\right) \to f_\sharp(A') \otimes_X B,$$

    which are natural on $(A, B) \in \mathrm{Ob}(\mathsf{H}(X))^2$ and $(A', B') \in \mathrm{Ob}(\mathsf{H}(Y))^2$ are invertible.
- Define the notion of the stable homotopy symmetric monoidal (resp. unital symmetric monoidal) 2-functor by making the evident changes.

[Ayoub, Def.4.1.57, Lem.4.1.58, p.440]

- A monoidal model category $(\mathfrak{M}, \otimes)$ is a model category equipped with a right and left closed monoidal structure (in the sennse of Def.2.1.119) satisfying the following aximo:

  (MMC) Let $f : A \to B$ and $g : U \to V$ be two cofibrations of $\mathfrak{M}$, Then the evivent morphism:

  $$f \square g : A \otimes V \coprod_{A \otimes U} B \otimes U \to B \otimes V$$

  is a cofibration which becomes a weak equivalence when f or g is a weak equivalence.

- We say that $\mathfrak{M}$ is symmetric when the underlying monoidal category is also equipped with a symmetry isomorphism.
- We say that $\mathfrak{M}$ is unital if the underlying monoidal ctegory is equipped with a unit which is cofibrant.
- For A a cofibrant object, the functors $A \otimes -$ and $- \otimes A$ are left Quillen functors. It follows from Lemma 4.1.26 that the bifuntor $- \otimes -$ preserves the weak equivalences between cofibrant objects. By Proposition 4.1.22, it admits a left derived functor

  $$- \otimes^{\mathsf{L}} -$$

  which makes $\mathbf{Ho}(\mathfrak{M})$ a left and right closed monoidal category.

- Let $\mathfrak{M}$ a unital symmetric monoidal model category. Denote by $\mathbf{1}$ the unit object of $\mathfrak{M}$. Suppose that $\mathfrak{M}$ is pointed. Then the functor Σ^1 is canonically isomorphic to $(\Sigma^1 \mathbf{1}) \otimes^{\mathsf{L}} -$. Furthermore, the permutation of factors:

  $$\tau : (\Sigma^1 \mathbf{1}) \otimes^{\mathsf{L}} (\Sigma^1 \mathbf{1}) \to (\Sigma^1 \mathbf{1}) \otimes^{\mathsf{L}} (\Sigma^1 \mathbf{1})$$

  is equal to the inverse of the commutative cogroup $\Sigma^2 \mathbf{1}$ module the identifiction $\Sigma^1(\Sigma^1 \mathbf{1}) \simeq \Sigma^1 \mathbf{1} \otimes^{\mathsf{L}} \Sigma^1 \mathbf{1}$. If furthermore $\mathfrak{M}$ is stable and left proper, then $\mathbf{Ho}(\mathfrak{M})$ is a monoidal triangulated category in the sense of Definition 2.1.148.

3.3 Review of general model category theory

We first review some basic of the model category, including the Bousfield localisation, which requires the cardinal consideration.

[Ayoub, Def.4.1.3, p.420; p.421]

- A model category $\mathfrak{M}$ is called left proper (resp. right proper) when weak equivalences of $\mathfrak{M}$ are stable by push-out by cofibrations (resp. stable by pull-back by fibrations). We say it is proper if it is left proper and right proper.
- A model category $\mathfrak{M}$ is called pointed if the unique morphism $\emptyset \to *$ between an initial object $\emptyset$ and a final object $*$

[Ayoub, Def.4.1.10, Def.4.1.11, Prop.4.1.12, p.422]

- Let X be an object of $\mathfrak{M}$.
  - A <u>cylinder</u> (C_X, p, i_0, i_1) on X is a commutative diagram in $\mathfrak{M}$:

$$\begin{tikzcd} X \arrow[dr, "i_0"'] \arrow[drr, equal, bend left] & & \\ & C_X \arrow[r, "p"] & X \\ X \arrow[ur, "i_1"] \arrow[urr, equal, bend right] & & \end{tikzcd}$$

    such that p is a weak equivalence and $i_0 \coprod i_1 : X \coprod X \to C_X$ is a cofibration.
  - A <u>path space</u> (P_X, c, e_0, e_1) on X is a commutative diagram in $\mathfrak{M}$:

$$\begin{tikzcd} X & & \\ & P_X \arrow[ul, "e_0"'] \arrow[dl, "e_1"] & X \arrow[l, "c"'] \arrow[ull, equal, bend right] \arrow[dll, equal, bend left] \\ X & & \end{tikzcd}$$

    such that c is a weak equivalence and $p_0 \times p_1 : P_X \to X \times X$ is a fibration.
- (<u>homotopy relations</u>)
  1. Let $f_0, f_1 : X \to Y$ be two arrows of $\mathcal{M}$. We say that <u>f_0 is left homotopic to f_1 relative to the cylinder (C_X, p, i_0, i_1)</u> if there exists an arrow $h : CX \to Y$ such that $f_0 = h \circ i_0$ and $f_1 = h \circ i_1$. The arrow h is called a <u>homotopy of f_0 and f_1 relative to the cylinder C_X</u> . We say that <u>f_0 is left homotopic to f_1</u> if there exists a cylinder relative to which f_0 is left homotopic to f_1.
  2. Dually, we hae the notion of <u>right homotopic</u> obtained using the path spaces.
- Let $f, g : X \to Y$ be two arrows of $\mathfrak{M}$ with X cofibrant and Y fibrant. Then the following assertions are equivalent:
  - f and g are left homotopic.
  - f and g are left homotopic relative to a fixed cylinder.
  - f and g are right homotopic.
  - f and g are right homotopic relative to a fixed path space.

  We denote by $\pi_0(X, Y)$ the quotient of $\hom_{\mathfrak{M}}(X, Y)$ by the homotopy relation.

[Ayoub, Def.4.1.13, p.423]

- Let X and Y be two objects of $\mathfrak{M}$ and $f, g : X \to Y$ two arrows. We suppose given the following data:
  - a left homotopy $h_g : C_X \to Y$ from f to g relative to a cylinder $'C_X, p, i_0, i_1)$.
  - a right homotopy $h_d : X \to P_Y$ fro mh t g relative to a path space (P_Y, c, e_0, e_1).

  A <u>correspondence</u> between h_g and h_d is an arrow $t : C_X \to P_Y$ making the following square commutative::

$$\begin{tikzcd} X \coprod X \arrow[r, "h_d \cup c o g"] \arrow[d, "i_0 \cup i_1"'] & P_Y \arrow[d, "(e_0, e_1)"] \\ C_X \arrow[r, "(h_g, g o p)"'] \arrow[ur, "t"] & Y \times Y \end{tikzcd}$$

  h_g and h_d are <u>corresponding</u> if a correspondence t exists.

[Ayoub, Th.4.1.17 p.424; Rem.4.1.18, p.425]

- The category

$$\mathbf{Ho}(\mathfrak{M}) = \mathfrak{M}\left[\mathbf{W}^{-1}\right]$$

exists and is equivalent to $\pi_0\mathfrak{M}_{cf}$. It is called the <u>homotopy category associated to the model category</u> $\mathfrak{M}$. Furthermore, for A cofibrant and X fibrant, we have a canonical isomorphism

$$\hom_{\mathbf{Ho}(\mathfrak{M})}(A, X) \simeq \pi_0(A, X).$$

- The proof of the theorem shows that the categories $\mathfrak{M}_c[\mathbf{W}^{-1}]$ and $\mathfrak{M}_f[\mathbf{W}^{-1}]$ exist and that they are equivalent to $\pi_0\mathfrak{M}_{cf}$ and so to $\mathbf{Ho}(\mathfrak{M})$. More precisely, the choices $R(X), Q(X), R(f)$ and $Q(f)$ define the functors

$$R : \mathfrak{M}_c \to \pi_0\mathfrak{M}_{cf}, \quad \text{and} \quad Q : \mathfrak{M}_f \to \pi_0\mathfrak{M}_{cf}$$

which send the weak equivalences to the isomorphisms. These induce the functors:

$$\mathfrak{M}_c[\mathbf{W}^{-1}] \to \pi_0\mathfrak{M}_{cf} \quad \text{and} \quad \mathfrak{M}_f[\mathbf{W}^{-1}] \to \pi_0\mathfrak{M}cf$$

which are equialences of categories.

[Ayoub, Def.4.1.21, Prop.4.1.22, p.426; Prop.4.1.23, Def.4.1.24, Prop.4.1.27, Def.4.1.28, p.427]

- (Derived functors)

1. Let $F : \mathfrak{M} \to \mathfrak{N}$ be a functor. We say that F is right derivable if there exists a pair $(\mathsf{R}F, \gamma)$ formed by
   - a functor $\mathsf{R}F : \mathbf{Ho}(\mathfrak{M}) \to \mathbf{Ho}(\mathfrak{N})$,
   - a square

$$\begin{tikzcd} \mathfrak{M} \arrow[r] \arrow[d, "F"'] \arrow[dr, phantom, "\gamma\nearrow"] & \mathbf{Ho}(\mathfrak{M}) \arrow[d, "\mathsf{R}F"] \\ \mathfrak{N} \arrow[r] & \mathbf{Ho}(\mathfrak{N}) \end{tikzcd}$$

   which is universal in the following sense. For any pair (T, β) formed by a functor at the level of homotopy categories and a square diagram as above, there exists a uniqute natural transformation $\alpha : \mathsf{R}F \to T$ such that $\beta = \alpha \circ \gamma$. The functor $\mathsf{R}F$ is called the right derived functor of F.

2. We obtain the notion of the left derived functor by duality. We denote by $\mathsf{L}F$ the left derived functor (when it exists) of F.

- Let $\mathfrak{M} \to \mathfrak{N}$ be a functor which preserve the weak equivalences between cofibrant (resp. fibrant) objects. Then F is left (resp. right) derivable.
- Let $(F, G) : \mathfrak{M} \to \mathfrak{N}$ be a an adjoint functor pair. Then the following assertions are equivalent:
  - F preserves the cofibrations and the trivial cofibrations.
  - G preserves the fibrations and trival fibrations.
  - F preserves the cofibrations and G preseres fibrations.
  - F preserves the trivial cofibrations and G preserves the trivial fibrations.
- The adjoint functor pair $(F, G) : \mathfrak{M} \to \mathfrak{N}$ verifying one of the equaivalent conditions above is called a Quillen adjunction . We say also that F (resp. G) is a left (resp. right) Quillen functor .
- Let $(F, G) : \mathfrak{M} \to \mathfrak{N}$ be a Quillen adjunction. Then F admit a left derived functor $\mathsf{L}F : \mathbf{Ho}(\mathfrak{M}) \to \mathbf{Ho}(\mathfrak{N})$. Dually, G admits a right derived functor $\mathsf{R}G : \mathbf{Ho}(\mathfrak{N}) \to \mathbf{Ho}(\mathfrak{M})$. Furthermore, the functor $\mathsf{L}F$ is naturally a left adjoint of $\mathsf{R}G$.
- A Quillen adjunction (F, G) is called a Quillen equivalence when $\mathsf{L}F$ (or $\mathsf{R}G$) is an equivalence of categories.

[Ayoub, Def.4.1.29, Def.4.1.30, Prop.4.1.31, p.428]

- Suppose given an adjunction $(F, G) : \mathfrak{M} \to \mathfrak{N}$ with $\mathfrak{M}, \mathfrak{N}$ model categories such that G sends weak equivalences between fibrant objects to weak equivalences , Then an object A of $\mathfrak{M}$ is called F-admissible , if $F(A)$ is cofibrant and that for any fibrant object X of $\mathfrak{N}$, the canonical morphism

$$\pi_0 \left(F(A), X\right) \to \hom_{\mathbf{Ho}(\mathcal{M})} \left(A, G(X)\right)$$

is invertible.
- An adjunction $(F, G) : \mathfrak{M} \to \mathfrak{N}$ with $\mathfrak{M}, \mathfrak{N}$ model categories is called Morel-Voevodsky adjunction when the following conditions are satified:
  - G sends weak equivalences between fibrant objects to weak equivalences.
  - For any object A of $\mathfrak{M}$, there exists a weak equivalnce $B \to A$ with B an F-admissible object. In this case, we say there exist enough F-admissibles .

Let $(F, G) : \mathfrak{M} \to \mathfrak{N}$ be a Morel-Voevodsky adjunction. Then F admits a left derived functor $\mathbf{L}F$ which is naturally left adjoint to $\mathbf{R}G$.

[Ayoub, Def.4.1.32, p.428; Def.4.1.34, Lem.4.1.35, p.429; p.430; Lem.4.1.36, p.430]

Let $f, g : A \to X$ be two arrows of $\mathfrak{M}$ with A cofibrant and X fibrant.

- Let (C_A, p, i_0, i_1) and (C'_A, p', i'_0, i'_1) be two cylinders over A. Two left homotopies

$$h : C_A \to X, \quad h' : C'_A \to X$$

of f and g are called left 2-homotopic if there exists a commutative diagram:

$$\begin{tikzcd}
\left(C_A \coprod_{A \coprod A} C'_A\right) \arrow[rr, "h \cup h'"] \arrow[d, "p \cup p'"'] \arrow[dr, "a"] & & X \\
A & D \arrow[l, "q"', "\simeq"] \arrow[ur, "l"'] &
\end{tikzcd}$$

with a a cofibration and q a weak equivalence The arrow l is called a 2-homotopy .

- The dual notion of right 2-homotopy of f and g is also defined.
- Denote by $\pi_1^g(A, X, f, g)$ (resp. $\pi_1^d(A, X, f, g)$ the class of left (resp. right) homotopies modulo the relation of 2-homotopy.
- Given a cylinder $(C_A, p, i_0.i_1)$ over A (resp. a loop space (P_X, c, e_0, e_1) over X), we denote by

$$\pi_1(A, X, f, g, C_A) \subset \pi_1^g(A, X, f, g)$$
$$(resp. \quad \pi_1(A, X, f, g, P_X) \subset \pi_1^d(A, X, f, g))$$

the subset of the left (resp. right) homotopies relative to C_A (resp. P_X) up to 2-homotopy.

- Under the preceeding hypotheses and notations,
  1. There exist a canonical isomorphism

  $$\pi_1^g(A, X, f, g) \simeq \pi_1^d(A, X, f, g)$$

  which associate to a left homotopy $h : C_A \to X$ a right homotopy $k : A \to P_X$ such that h and k are correnpondants in th esense of Definition 4.1.13.
  2. The inclusion $\pi_1(A, X, f, g, C_A) \subset \pi_1^g(A, X, f, g)$ is bijective. In particular, $\pi_1^g(A, X, f, g)$ and $\pi_1^d(A, X, f, g)$ are sets.
- Denote by $\pi_1(A, X, f, g)$ the one of the two sets canonically isomorphic to $\pi_1^g(A, X, f, g)$ or $\pi_1^d(A, X, f, g)$. The next discussion show that the set $\pi_1(A, X, f, g)$ is bifunctorial in A and X :
- Let $u : A' \to A$ be an arrow between cofibrant objects of $\mathfrak{M}$. The composite:

$$\pi_1^g(A, X, f, g) \xleftarrow{\sim} \pi_1(A, X, f, g, C_A) \to \pi_1(A', X, f \circ u, g \circ u, C'_A) \xrightarrow{\sim} \pi_1^g(A', X, f \circ u, g \circ u)$$

does not depend on a choice of commutative diagram:

$$\begin{tikzcd}
A' \coprod A' \arrow[r, "i'_0 \cup i'_1"] \arrow[d] & C_{A'} \arrow[r, "p'"] \arrow[d, "v"] & A' \arrow[d, "u"] \\
A \coprod A \arrow[r, "i_0 \cup i_1"] & C_A \arrow[r, "p"] & A
\end{tikzcd}$$

Furthermore, we have a commutative diagram:

$$\begin{tikzcd}
\pi_1^g(A, X, f, g) \arrow[r] \arrow[d, "\sim"'] & \pi_1^g(A', X, f \circ u, g \circ u) \arrow[d, "\sim"] \\
\pi_1^d(A, X, f, g) \arrow[r] & \pi_1^d(A', X, f \circ u, g \circ u)
\end{tikzcd}$$

[Ayoub, Th.4.1.38 Def.4.1.39, p.431; Def.4.1.44, p.433; Th.4.1.49, p.434; Lem.4.1.51, p.436; Th.4.1.56 p.440]

- If the model category $\mathfrak{M}$ is pointed. Then the bifunctor

$$\pi_1(-,-) : \mathfrak{M}_c^{\mathrm{op}} \times \mathfrak{M}_f \to \mathit{Sets}$$

induces a bifunctor

$$\pi_1(-,-) : \mathbf{Ho}(\mathfrak{M})^{\mathrm{op}} \times \mathbf{Ho}(\mathfrak{M}) \to \mathit{Sets}$$

Furthermore, there exists a couple of adjoint functors

$$(\Sigma^1, \Omega^1) : \mathbf{Ho}(\mathfrak{M}) \to \mathbf{Ho}(\mathfrak{M})$$

and the functorial isomorphisms on cofibrant A and fibrant X:

$$\hom_{\mathbf{Ho}(\mathfrak{M})}(\Sigma^1 A, X) \simeq \pi_1(A, X) \simeq \hom_{\mathbf{Ho}(\mathfrak{M})}(A, \Omega^1 X)$$

Finally, for cofibrant A, the object $\Sigma^1 A$ is canonically isomorphic (in $\mathbf{Ho}(\mathfrak{M})$) to the push-out diagrams:

$$\begin{tikzcd} A \coprod A \arrow[r] \arrow[d] & C_A \\ * & \end{tikzcd}$$

The dual statement is obviously true for $\Omega^1 X$.

- For a pointed model category $\mathfrak{M}$, the endofunctor Σ^1 of $\mathbf{Ho}(\mathfrak{M})$ is called the <u>suspension functor</u> . Its adjoint Ω^1 is called the <u>cosuspension functor</u> .
- A model category $\mathfrak{M}$ is called <u>stable</u> if the category $\mathfrak{M}$ is pointed and if the suspension functor Σ^1 is an autoequivalence of $\mathbf{Ho}(\mathfrak{M})$.
- Suppose that the model category $\mathfrak{M}$ is stable and left proper. Then $\mathbf{Ho}(\mathfrak{M})$ is naturally a traiangulated category, where the suspension functor is given by Σ^1 and the distinguished triangles are given by the triangles of cofibrations.
- Suppose a Quillen adjunction between two model categories which are stable and left proper is given:

$$(F, G) : \mathfrak{M} \to \mathfrak{N}$$

Then, $\mathsf{L}F$ and $\mathsf{R}G$ are triangulated functors.

- Let $\mathfrak{M}$ be a left proper stable model category . Then a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is homotopy cocartesian if and only if it is homotopy cartesian.

[Ayoub, Def.2.1.119, p.230; Def.4.1.57, Lem.4.1.58, p.440]

- Let $(\mathcal{C},\otimes)$ be a monoidal category. We say it is left closed if for any object A of $\mathcal{C}$, the functor $A\otimes -$ admits a right adjoint. We say that $\mathcal{C}$ is right closed if for any object A of $\mathcal{C}$, the functor $-\otimes A$ admits a right adjoint.
- A monoidal category $\mathcal{C}$ is right closed if the $\otimes$-opposed category $\mathcal{C}^0$ is left closed and vice versa. So just to study one type of the closed monoidal categories.

  Thereafter, we consider mainly the right closed monnoidal categories. We denote by $\underline{\mathrm{Hom}}(A,-)$ the right adjoint of $-\otimes A$. There is thus isomorphisms:

  $$\mathrm{hom}_{\mathcal{C}}(U\otimes A,V)\xrightarrow{\sim}\mathrm{hom}_{\mathcal{C}}(U,\underline{\mathrm{Hom}}(A,V))$$

  as well as the arrows:

  $$ev:\underline{\mathrm{Hom}}(A,v)\otimes A\to V\quad\text{and}\quad\delta:U\to\underline{\mathrm{Hom}}(A,U\otimes A)$$

  natural in U and V of $\mathcal{C}$.
- When we will need to cisider right and left closed monoidal categories, we denote, to distinguish, $\underline{\mathrm{Hom}}_g(A,-)$ and $\underline{\mathrm{Hom}}_d(A,-)$ the respective right adjoints of $A\otimes -$ and $-\otimes A$.
- A monoidal model category $(\mathfrak{M},\otimes)$ is a model category equipped with a right and left closed monoidal structure (in the sennse of Def.2.1.119) satisfying the following aximo:

  (MMC) Let $f:A\to B$ and $g:U\to V$ be two cofibrations of $\mathfrak{M}$, Then the evivent morphism:

  $$f\square g:A\otimes V\coprod_{A\otimes U}B\otimes U\to B\otimes V$$

  is a cofibration which becomes a weak equivalence when f or g is a weak equivalence.
- We say that $\mathfrak{M}$ is symmetric when the underlying monoidal category is also equipped with a symmetry isomorphism.
- We say that $\mathfrak{M}$ is unital if the underlying monoidal ctegory is equipped with a unit which is cofibrant.
- For A a cofibrant object, the functors $A\otimes -$ and $-\otimes A$ are left Quillen functors. It follows from Lemma 4.1.26 that the bifuntor $-\otimes-$ preserves the weak equivalences between cofibrant objects. By Proposition 4.1.22, it admits a left derived functor

  $$-\otimes^{\mathsf{L}}-$$

  which makes $\mathbf{Ho}(\mathfrak{M})$ a left and right closed monoidal category.
- Let $\mathfrak{M}$ a unital symmetric monoidal model category. Denote by $\mathbf{1}$ the unit object of $\mathfrak{M}$. Suppose that $\mathfrak{M}$ is pointed. Then the functor Σ^1 is canonically isomorphic to $(\Sigma^1\mathbf{1})\otimes^{\mathsf{L}}-$. Furthermore, the permutation of factors:

  $$\tau:(\Sigma^1\mathbf{1})\otimes^{\mathsf{L}}(\Sigma^1\mathbf{1})\to(\Sigma^1\mathbf{1})\otimes^{\mathsf{L}}(\Sigma^1\mathbf{1})$$

  is equal to the inverse of the commutative cogroup $\Sigma^2\mathbf{1}$ module the identifiction $\Sigma^1(\Sigma^1\mathbf{1})\simeq\Sigma^1\mathbf{1}\otimes^{\mathsf{L}}\Sigma^1\mathbf{1}$. If furthermore $\mathfrak{M}$ is stable and left proper, then $\mathbf{Ho}(\mathfrak{M})$ is a monoidal triangulated category in the sense of Definition 2.1.148.

We now take into account cardinal considerations in order to construct the Bousfield localisation.

[Ayoub, 4.2.1, Def. 4.2.1, p..442; Def. 4.2.5, p..443; p.444]

- We say that a category $\mathcal{K}$ is pseudo-doscrete if there exists two discrete full sub-categories $\mathcal{J}_0$ and $\mathcal{J}_1$ such that
$$\hom(j_1, j_0) = \emptyset$$
for all $(j_0, j_1) \in \mathrm{Ob}(\mathcal{J}_0) \times \mathrm{Ob}(\mathcal{J}_1)$.
- By the cardinal of a small category $\mathcal{K}$, we mean the cardinal of the set $\mathbf{Ar}(\mathcal{K})$.
- Let α be an ordinal (not necessarily infinite). A category $\mathcal{I}$ is called α-filtered if it is non empty and if for any functor $P : \mathcal{J} \to \mathcal{I}$ with $\mathcal{J}$ pseudo-discrete and whose cardinal inferior or equal to α, there exists $i \in \mathcal{I}$ such that the set
$$\mathbf{Lim}_{j \in \mathrm{Ob}(\mathcal{J})} \hom_{\mathcal{I}}(P(j), i)$$
is non emptry.
- A functor $F : \mathcal{A} \to \mathcal{B}$ between categories admitting small colimits is called α-accessible if its commutes with with colimits with respect to α-filtered small categories.
- An object $A \in \mathrm{Ob}(\mathcal{A})$ is called α-accessible if the functor $\hom_{\mathcal{A}}(A, -)$ is α-accessible.
- Recall that a monomorphism in a category $\mathcal{C}$ is an arrow α such that the map $\hom(X, a)$ is injective for any object X.
- When $\mathcal{C}$ admits finite colimits, an arrow whose arbitray pushout is a monomorphism is called a universal monomorphism.

[Ayoub, Def. 4.2.11, Def. 4.2.13, p.445; Def. 4.2.16, p.446; Prop.4.2.20, Prop.4.2.21, p.447]

- Let B be an object of a category $\mathcal{C}$. Denote by

$$\mathbf{Sub}(B) \subset \mathcal{C}/B$$

the full subcategory whose objects are monomorphisms $\alpha : A \to B$. Sometimes, A is called a sub-object of B.

- Suppose $\mathcal{C}$ has small colimits and α an ordinal. Denote by $\mathbf{Sub}_\alpha(B)$ the full subcategory formed by monomorphisms with α-accessible sources.

- Let $\mathcal{C}$ be a category with small colimits. We say an object X of $\mathcal{C}$ is the α-filtered colimit of its α-accessible sub-objects, if the following two conditions are satisfied:

  - $\mathbf{Sub}_\alpha(X)$ is essentially small and α-filtered.
  - The canonical arrow $\left(\mathrm{Colim}_{(A\to X)\in \mathbf{Sub}_\alpha(X)} A\right) \to X$ is invertible.

- Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is α-presentable if the following conditions are verified with any cardinal β equal or larger than α:

  - $\mathcal{C}$ is bicomplete. The filtere colimits commute with finite limits. Moreover, the β-filtered colimits commute with limits over the category whose cardinal is equal or smaller than β.
  - The monomorphisms iin $\mathcal{C}$ are universal.
  - Any object of $\mathcal{C}$ is accessible. A sub-object of a β-accessible object is again β-accesible.
  - Any object of $\mathcal{C}$ is the β-filtered colimit of its β-accessible sub-objects.
  - The sub-category $\mathcal{C}_\beta$ formed by β-accessible objects is essentially small.

  When $\mathcal{C}$ is α-presentable with α finite and larger or equal to 5, we write that $\mathcal{C}$ is finitely presentable.

  We say $\mathcal{C}$ is presentable when it is α-presentable for some cardinal α.

- Let $\mathcal{I}$ be a small category and $\mathcal{C}$ a presentable category. Then $\underline{\mathrm{HOM}}(\mathcal{I}, \mathcal{C})$ is still presentable.

- Let $\mathcal{C}$ be a α-presentable category and $\mathcal{D}$ a bicomplete category. We suppose given an adjunction $(F, G) : \mathcal{C} \to \mathcal{D}$ such that:

  - G is fully faithful and α-accessible,
  - F commuts with finite limits,
  - the functor $G \circ F$ preserve β-acdessible objects for β larger or equal to α.

  Then $\mathcal{D}$ is α-presentable.

[Ayoub, Def.4.2.23, Def.4.2.24, p.448; Prop.4.2.26, p.449]

- Let λ be an ordinal. We call λ-sequeence a functor $A : \lambda \to \mathcal{C}$ which we schematically have by:

$$A_0 \to A_1 \to \cdots \to A_\nu \to A_{\nu+1} \to \cdots$$

and which commutes with colimits (when they exist), i.e. for any $\nu \in \lambda$ a limit ordinal, we have $A_\nu \simeq \mathrm{Colim}_{\mu \in \nu} A_\mu$. The arrow $A_0 \to \mathrm{Colim}_{\nu \in \lambda} A_\nu$ is called the transfinite composition of the Λ-sequence A.

- let $\mathcal{C}$ be a category admitting the small colimits and $F \subset \mathbf{Ar}(\mathcal{C})$ a class of arrows. We denote by $\mathsf{Cell}(F) \subset \mathbf{Ar}(\mathcal{C})$ the maximam small classes containing F and stable by pushout and transfinite composition.

- Let $\mathcal{C}$ be a category admitting small colimits. Let $F \subset \mathbf{Ar}(\mathcal{C})$ be a set of arrows whose sources and targets α-accessible for a fixed cardinal α. It then exists a functor

$$\Psi_{F,\alpha} : \underline{\mathbf{HOM}}(\underline{\mathbf{1}}, \mathcal{C}) \to \underline{\mathbf{HOM}}(\underline{\mathbf{2}}, \mathcal{C})$$

which to an arrow $f : U \to V$ associates a factorisation:

$$\begin{tikzcd} U \arrow[r] \arrow[rr, bend left, "f"] & \Phi_{F,\alpha}(f) \arrow[r] & V \end{tikzcd}$$

which satisfies the following properties:

1. The arrow $U \to \Phi_{F,\alpha}(f)$ is in $\mathsf{Cell}(F)$.
2. The arrow $\Phi_{F,\alpha}(f) \to V$ is in $\mathsf{RLP}(F)$.
3. The functor $\Phi_{F,\alpha}(f)$ is α-accessible.
4. Let β be an infinite cardinal verifying the two conditions:

(a) β is strictly superior to α and superior or equal to the cardinal of the set F.

(b) for any α-accessible object A and any β-accessible object X of $\mathcal{C}$, the set $\hom_{\mathcal{C}}(A, X)$ is of cardinal inferior or equal to β.

Then, the object $\Phi_{F,\alpha}(f)$ is β-accessible when U and V are β-accessible.

[Ayoub, Def,4.2.39, p.456]

We say that a model category $(\mathfrak{M}, \mathbf{W}, \mathbf{Cof}, \mathbf{Fib})$ is α-presentable by cofibrations if the following conditions are satisfied:

- $\mathfrak{M}$ is α-presentable, as an abstruct category, in the sense of the Definition 4.2.16, p.446.
- The cofibrations of $\mathfrak{M}$ are monomorphisms.
- Denote by $\mathbf{Cof}_\alpha$ the class of cofibrations whose target is α-accessible. Then

$$\mathbf{Fib} = \mathrm{RLP}\,(\mathbf{Cof}_\alpha \cap \mathbf{W})\,; \quad \mathbf{Fib} \cap \mathbf{W} = \mathrm{RLP}\,(\mathbf{Cof}_\alpha)$$

The cardinal α is called the essential size of $\mathfrak{M}$.

[Ayoub, Def. 4.2.58, Rem.4.2.59, p.462]

- Let $\mathcal{A}$ be a sub-class of $\mathbf{Ar}\,(\mathbf{Ho}(\mathfrak{M}))$. A left Bousfield localisation with respect to $\mathcal{A}$ is a model category $(\mathsf{L}_{\mathcal{A}}\mathfrak{M}, \mathbf{W}_{\mathcal{A}}, \mathbf{Cof}_{\mathcal{A}}, \mathbf{Fib}_{\mathcal{A}})$ equipped with a Quillen adjunction

$$(U_{\mathcal{A}}, V_{\mathcal{A}}) : \mathfrak{M} \to \mathsf{L}_{\mathcal{A}}\mathfrak{M}$$

such that $\mathsf{L}U_{\mathcal{A}}(f)$ is invertible for any $f \in \mathcal{A}$ and the following universal property is satisfied: For any model category $\mathfrak{N}$ equipped with a Quillen adjunction $(F, G) : \mathfrak{M} \to \mathfrak{N}$ such that $\mathsf{L}F(f)$ is invertible for $f \in \mathcal{A}$, there exists a unique Quillen adjunction $(F_{\mathcal{A}}, G_{\mathcal{A}})$ makeing the following diagram commutative:

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "F"] \arrow[d, "U_{\mathcal{A}}"'] & \mathfrak{N} \\ \mathsf{L}_{\mathcal{A}}\mathfrak{M} \arrow[ur, "F_{\mathcal{A}}"'] & \end{tikzcd}$$

- Let $(F, G) : \mathfrak{M} \to \mathfrak{N}$ be a Quillen adjunction. For two classes of morphisms:

$$\mathcal{A} \subset \mathsf{Ar}\,(\mathbf{Ho}(\mathfrak{M}))\,, \quad \mathcal{B} \subset \mathsf{Ar}\,(\mathbf{Ho}(\mathfrak{N}))$$

we suppose that

$$\mathsf{L}F(\mathcal{A}) \subset \mathcal{B}.$$

Then the adjunction (F, G) induces naturally a Quillen adjunction

$$(F, G) : \mathsf{L}_{\mathcal{A}}\mathfrak{M} \to \mathsf{L}_{\mathcal{B}}\mathfrak{N}$$

when the Bousfield localisations exist.

[Ayoub, Lem. 4.2.60, Lem. 4.2.61, p.464]

In a model category $\mathfrak{M}$, let $a : A \to B$ be a cofibration between cofibrant objects and X be a fibrant object. Then the following conditions are equivalent:

- The morphism
$$\pi_0(a, X) : \pi_0(B, X) \to \pi_0(A, X)$$
is surjective (resp. and injective).
- The arrow $X \to *$ admits a right lifting property concerning
$$a : A \to B$$
(resp. and
$\left[\mathrm{Cyl}_a(A) \coprod_{(A \coprod A)} (B \coprod B)\right] \to \mathrm{Cyl}_a(B)$).

[Ayoub, Def. 4.2.63, p.464]

For any cofibration $a : A \to B$, we choose a cofibration $\nabla(a)$:

$$\left[\mathrm{Cyl}_a(A) \coprod_{(A \coprod A)} (B \coprod B)\right] \to \mathrm{Cyl}_a(B)$$

with $\mathrm{Cyl}_a(A)$ and $\mathrm{Cyl}_a(B)$ the cylynders of A and B.
Fo a class F of cofibrations, we set

$$\nabla(F) = \{\nabla(a) \mid a \in F\}$$
$$\nabla_n(F) = \nabla\left(\nabla_{n-1}(F)\right)$$
$$\nabla_\infty(F) = \cup_{n \in \mathbb{N}} \nabla_n(F)$$

[Ayoub, Def. 4.2.64, p.465; Lem.4.2.69, p.467]

- Fix a sub-class $\mathcal{A} \subset \mathbf{Ar}\left(\mathbf{Ho}(\mathfrak{M})\right)$. Let $\underline{\mathcal{A}} \subset \mathbf{Ar}(\mathfrak{M})$ be a up to an isomorphism lift of $\mathcal{A}$, consisting of cofibrations between cofibrant objects. When $\mathcal{A}$ is essentialy small, we implicity suppose $\underline{\mathcal{A}}$ is a set. Then, set

$$\nabla_\infty(\mathcal{A}) := \mathrm{Im}\left(\nabla_\infty(\underline{\mathcal{A}}) \to \mathbf{Ho}(\mathfrak{M})\right),$$

which is shown to be independent of a choice of $\underline{\mathcal{A}}$ (by Lemma 4.2.62, p.463).

  - An object X of $\mathfrak{M}$ is called <u>$\mathcal{A}$-local</u> when for any arrow $\alpha; A \to B$ of $\nabla_\infty(\mathcal{A})$, the morphsim

$$\hom(f, X) : \hom_{\mathbf{Ho}(\mathfrak{M})}(B, X) \to \hom_{\mathbf{Ho}(\mathfrak{M})}(A, X)$$

    is invertible.
  - We denote $\mathfrak{M}_{\mathcal{A}-loc}$ (resp. $\mathbf{Ho}(\mathfrak{M})_{\mathcal{A}-loc}$) the full subcategory of $\mathfrak{M}$ (resp. $\mathbf{Ho}(\mathfrak{M})$) made of $\mathcal{A}$-local objects.
  - An arrow $f : A \to B$ is called <u>$\mathcal{A}$-weak equivalence</u> when for any $\mathcal{A}$-local object X,, the morphsim

$$\hom(f, X) : \hom_{\mathbf{Ho}(\mathfrak{M})}(B, X) \to \hom_{\mathbf{Ho}(\mathfrak{M})}(A, X)$$

    is invertible.
  - Denote by $\mathbf{W}_{\mathcal{A}}$ the class of $\mathcal{A}$-weak equivalences.
  - Set $\mathbf{Fib}_{\mathcal{A}} := \mathrm{RLP}\left(\mathbf{Cof} \cap \mathbf{W}_{\mathcal{A}}\right)$.

- Let a morphism of λ-sequences:

$$\begin{tikzcd}
A'_0 \arrow[r, "f'_0"] \arrow[d] & A'_1 \arrow[r, "f'_1"] \arrow[d] & \cdots \arrow[r] & A'_\nu \arrow[r, "f'_\nu"] \arrow[d] & A'_{\nu+1} \arrow[r] \arrow[d] & \cdots \\
A_0 \arrow[r, "f_0"] & A_1 \arrow[r, "f_1"] & \cdots \arrow[r] & A_\nu \arrow[r, "f_\nu"] & A_{\nu+1} \arrow[r] & \cdots
\end{tikzcd}$$

in the left proper model category $\mathfrak{M}$. We suppose that the vertical arrows are weak equivalences and the horizontal arrows are cofibrations. Then the evident morphism

$$\mathrm{Colim}_{\nu \in \lambda} A'_\nu \to \mathrm{Colim}_{\nu \in \lambda} A_\nu$$

is a weak equivalence.

[Ayoub, Th.4.2.71, p.468; Prop.4.2.76, p.472; Prop.4.2.82, p.474]

- Suppose the model category $\mathfrak{M}$ is left proper and presentable by cofibrations. If the class $\mathcal{A}$ is essentially small, then the quadrupplet

$$(\mathfrak{M}, \mathbf{W}_{\mathcal{A}}, \mathbf{Cof}, \mathbf{Fib}_{\mathcal{A}})$$

is a model category, which is still left proper and presentable by cofibrations. Moreover, this is the left Bousfield localisation with respect to $\mathcal{A}$.

- The model category $(\mathfrak{M}, \mathbf{W}_{\mathcal{A}}, \mathbf{Cof}, \mathbf{Fib}_{\mathcal{A}})$ is a monoidal model category (see Definition 4.1.57) when the following conditions are satisfied:
  - for $A \in \mathrm{Ob}(\mathfrak{M})$ cofibrant and $f \in \underline{\mathcal{A}}$, the arrows $A \otimes f$ and $f \otimes A$ are $\mathcal{A}$-weak equivalences.
  - The model category $(\mathfrak{M}, \mathbf{W}_{\mathcal{A}}, \mathbf{Cof}, \mathbf{Fib}_{\mathcal{A}})$ is stable.
- Let $\mathfrak{M}$ be a model category which is presentable by cofibrations and $\mathcal{A}$ an essentially small class of arrows in $\mathbf{Ho}(\mathfrak{M})$. We suppose that $\mathfrak{M}$ is stable. So that the $\mathcal{A}$-localized structure of $\mathfrak{M}$ become still stable, it is necessary and sufficent that one of the following two equivalent conditions are satisfied:
  - the functor $\Omega^1 : \mathbf{Ho}(\mathfrak{M}) \to \mathbf{Ho}(\mathfrak{M})$ sends the arrows of $\nabla_{\infty}(\mathcal{A})$ to the $\mathcal{A}$-weak equivalances.
  - the functor Σ^1 preseres the $\mathcal{A}$-local objects of $\mathbf{Ho}(\mathfrak{M})$.

3.4 Review of general theory of symmetric spectra

[Ayoub, 4.3. p.474]

Let $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ be a unital graded monoid in the category of groups: $\forall (m,n) \in \mathbb{N}\times\mathbb{N}, \exists$ a group homomorphism

$$\phi_{m.n} : \Phi_m \times \Phi_n \to \Phi_{m+n}$$

satisfying the following conditions:

- $\forall (m,n,r) \in \mathbb{N}^3$, the folowing diagram is commutative:

$$\begin{tikzcd} \Phi_m \times \Phi_n \times \Phi_r \arrow[r, "\Phi_{m,n}\times I_{\Phi_r}"] \arrow[d, "I_{\Psi_m}\times\phi_{n,r}"'] & \Phi_{m+n} \times \Phi_r \arrow[d, "\phi_{m+n,r}"] \\ \Phi_m \times \Phi_{n+r} \arrow[r, "\phi_{m,n+r}"'] & \Phi_{m+n+r} \end{tikzcd}$$

- Let 1 be the unit of Φ_0. Then $\forall n \in \mathbb{N}$ the folowwing two compositions are identities:

$$\Phi_n \simeq \{1\} \times \Phi_n \to \Phi_0 \times \Phi_n \xrightarrow{\phi_{0,n}} \Phi_n$$
$$\Phi_n \simeq \Phi_n \times \{1\} \to \Phi_n \times \Phi_0 \xrightarrow{\phi_{n,0}} \Phi_n$$

[Ayoub, Ex.4.3.1, p.474]

Two important examples of unital graded monoids:

$\{1\} = (\{1\})_{n\in\mathbb{N}}$ The trivial unital graded monoid , given on each degree by the group of a single element $\{1\}$.

$\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ The unital graded monoid of symmetric groups , where the group morphism

$$\begin{aligned} \phi_{m,n} : \Sigma_m \times \Sigma_n &\to \Sigma_{m+n} \\ (f,g) &\mapsto f \bullet g \end{aligned}$$

is defined by

$$f \bullet g(i) = \begin{cases} f(i) & \text{if } 1 \le i \le m, \\ g(i-m)+m & \text{if } m+1 \le i \le m+n. \end{cases} \qquad (i \in \{1,\dots,m+n\}).$$

[Ayoub, Def,4.3.3, p.475]

Let $\mathcal{C}$ be a category.

- For a group G, denote by $\mathbf{Rep}(G, \mathcal{C})$ the category of G-representations in $\mathcal{C}$, i.e. the category of functors $\bullet\{G\} \to \mathcal{C}$ where $\bullet\{G\}$ is the category of a single object $\bullet$ with $\mathrm{end}(\bullet) = G$.
- Denote by $\mathbf{Suite}(\Phi, \mathcal{C})$ the category $\prod_{n\in\mathbb{N}} \mathbf{Rep}(\Phi_n, \mathcal{C})$. Ab object of this category is called a Φ-symmetric sequence in $\mathcal{C}$. This is a family of objects $(X_n)_{n\in\mathbb{N}}$ of $\mathcal{C}$ equipped with actions $\Phi_n \to \mathrm{Aut}(X_n)$ for $n \in \mathbb{N}$.

[Ayoub, Def,4.3.4, p.475]

Let $\mathcal{C}$ be a category. A Φ-symmetric endofunctor of $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{C}$ equippedwith an action $a_F : \Phi_n \to \mathrm{Aut}\,(F^{\circ n})$ for any $n \in \mathbb{N}$, such that the following diagram is commutative for any $(m, n) \in \mathbb{N} \times \mathbb{N}$:

$$\begin{tikzcd}
\Phi_m \times \Phi_n \arrow[r] \arrow[d, "\phi_{m,n}"'] & \mathrm{Aut}\left(\overbrace{F \circ \cdots \circ F}^{m}\right) \times \mathrm{Aut}\left(\overbrace{F \circ \cdots \circ F}^{n}\right) \arrow[d] \\
\Phi_{m+n} \arrow[r] & \mathrm{Aut}\left(\underbrace{F \circ \cdots \circ F}_{m} \circ \underbrace{F \circ \cdots \circ F}_{n}\right)
\end{tikzcd}$$

[Ayoub, Def,4.3.6, p.475]

Let F be a Φ-symmetric endofunctor of $\mathcal{C}$.

- A Φ-symmetric F-spectrum (or simply (F,Φ)-spectrum) $\mathbf{X}$ is a Φ-symmetric sequence $(\mathbf{X}_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ equipped with a assembly map $\gamma_n : F(\mathbf{X}_n) \to \mathbf{X}_{n+1}$ such that for any $(m,n) \in \mathbb{N}^2$ the composite

$$F^{\circ m}(\mathbf{X}_n) \xrightarrow{F^{\circ(m-1)}(\gamma_n)} F^{\circ(m-1)}(\mathbf{X}_{1+n}) \to \cdots$$
$$\to F(\mathbf{X}_{m-1+n}) \xrightarrow{\gamma_{m-1+n}} \mathbf{X}_{m+n}$$

  is $\Phi_m \times \Phi_n$-equivariant relative to:

  - The action on $F^{\circ m}(\mathbf{X}_n)$ is induced by the action of Φ_m on $F^{\circ m}$ and the action of Φ_n on $\mathbf{X}_n$.
  - The action on $\mathbf{X}_{m+n}$ is obtained by restricting the action of Φ_{m+n} via the morphism $\phi_{m,n}$: $\Phi_m \times \Phi_n \to \Phi_{m+n}$.

- A morphism of (F,Φ)-spectra from $\mathbf{X} = (\mathbf{X}_n)_{n\in\mathbb{N}}$ to $\mathbf{Y} = (\mathbf{Y}_n)_{n\in\mathbb{N}}$ is a morphism of Φ-symmetric sequences $\mathbf{X}_n \to \mathbf{Y}_n$ such that the following diagram is commutative for any $n \in \mathbb{N}$:

$$\begin{tikzcd} F\mathbf{X}_n \arrow[r] \arrow[d] & \mathbf{X}_{n+1} \arrow[d] \\ F\mathbf{Y}_n \arrow[r] & \mathbf{Y}_{n+1} \end{tikzcd}$$

  We denote by $\mathbf{Spect}_F^{\Phi}(\mathcal{C})$ the category of Φ-symmetric F-spectra.

- For $n \in \mathbb{N}$, define $\underline{\mathrm{Ev}}_n : \mathbf{Spect}_F^{\Phi}(\mathcal{C}) \to \mathbf{Rep}(\Phi_n, \mathcal{C})$ and $\mathrm{Ev}_n : \mathbf{Spect}_F^{\Phi}(\mathcal{C}) \to \mathcal{C}$ as follows:

$$\begin{cases} \underline{\mathrm{Ev}}_n : \mathbf{Spect}_F^{\Phi}(\mathcal{C}) & \to \mathbf{Rep}(\Phi_n, \mathcal{C}) \\ \mathbf{X} = (\mathbf{X}_n)_{n\in\mathbb{N}} & \mapsto \mathbf{X}_n \text{ with the } \Phi_n\text{-action} \end{cases}$$
$$\begin{cases} \mathrm{Ev}_n : \mathbf{Spect}_F^{\Phi}(\mathcal{C}) & \to \mathcal{C} \\ \mathbf{X} = (\mathbf{X}_n)_{n\in\mathbb{N}} & \mapsto \mathbf{X}_n \end{cases}$$

[Ayoub, Lem,4.3.9, Def,4.3.10, p.476]

- Suppose that a category $\mathcal{C}$ admits relevant colimits. Let $\alpha : H \to G$ be a group morphism.
  - Denote the evident restriction functor
$$\alpha_* : \mathbf{Rep}(G, \mathcal{C}) \to \mathbf{Rep}(H, \mathcal{C})$$
    by Oub^G_H.
  - It admits a left adjoint functor
$$\alpha^* : \mathbf{Rep}(H, \mathcal{C}) \to \mathbf{Rep}(G, \mathcal{C}),$$
    which is denoted by Ind^G_H.

  Let F be a Φ-symmetric endofunctor of $\mathcal{C}$, which commutes with relevant colimits.For $p \in \mathbb{N}$ the functor $\underline{\mathrm{Ev}}_p$ admits a left adjoint
$$\underline{\mathrm{Sus}}^p_{F,\Phi} : \mathbf{Rep}\left(\Phi_p, \mathcal{C}\right) \to \mathbf{Spect}^\Phi_F(\mathcal{C}).$$
  For $X \in \mathrm{Ob}\left(\mathbf{Rep}\left(\Phi_p, \mathcal{C}\right)\right)$, the (F,Φ)-spectrum $\underline{\mathrm{Sus}}^p_{F,\Phi}(X)$ is as follows:
  - As a Φ_n-representation,
$$\underline{\mathrm{Sus}}^p_{F,\Phi}(X)_n = \begin{cases} \mathrm{Ind}^{\Phi_n}_{\Phi_{n-p} \times \Phi_p}\left(F^{\circ(n-p)}(X)\right) & n \geqq p \\ \emptyset & n < p \end{cases}$$
  - For $n \geqq p$, the assemply morphism
$$F\left[\underline{\mathrm{Sus}}^p_{F,\Phi}(X)_n\right] \to \underline{\mathrm{Sus}}^p_{F,\Phi}(X)_{n+1}$$
    is the composite:
$$\begin{aligned} F\left(\mathrm{Ind}^{\Phi_n}_{\Phi_{n-p} \times \Phi_p}\left(F^{\circ(n-p)}(X)\right)\right) &\simeq \left(\mathrm{Ind}^{\{1\} \times \Phi_n}_{\{1\} \times \Phi_{n-p} \times \Phi_p}\left(F^{\circ(1+n-p)}(X)\right)\right) \\ &\to \left(\mathrm{Ind}^{\Phi_{1+n}}_{\Phi_{1+n-p} \times \Phi_p}\left(F^{\circ(1+n-p)}(X)\right)\right) \end{aligned}$$

- Under the same hypothesis as above , set
$$\begin{cases} \mathrm{Sus}^p_{F,\Phi} & = \underline{\mathrm{Sus}}^p_{F,\Phi} \circ \mathrm{Ind}^{\Phi_p}_1 \\ \mathrm{Ev}_p & = \mathrm{Oub}^{\Phi_p}_1 \circ \underline{\mathrm{Ev}}_p \end{cases}$$
  which forms an adjunction
$$\left(\mathrm{Sus}^p_{F,\Phi}, \mathrm{Ev}_p\right) : \mathcal{C} \rightleftarrows \mathbf{Rep}\left(\Phi_p, \mathcal{C}\right) \rightleftarrows \mathbf{Spect}^\Phi_F(\mathcal{C})$$
  For $X \in \mathrm{Ob}(\mathfrak{M})$, the (F,Φ)-spectrum $\mathrm{Sus}^p_{F,\Phi}(X)$ is called <u>p-th suspension (F,Φ)-spectrum</u> of X. At the level $n \geqq p$, this (F,Φ)-spectrum i given by $\mathrm{Ind}^{\Phi_n}_{\Phi_{n-p}} F^{\circ(n-p)}X$, where the induction is given by the group homomorphism
$$\phi_{n-p,p}(-,1); \Phi_{n-p} \to \Phi_n$$

[Ayoub, Def,4.3.13, p.478]

For the category of Φ-symmetric sequences in $\mathcal{C}$

$$\mathbf{Suite}(\Phi,\mathcal{C}) = \prod_{n\in\mathbb{N}} \mathbf{Rep}(\Phi_n,\mathcal{C}),$$

given in Definition 4.3.3,

- Denote by

$$\begin{aligned} s_- : \mathbf{Suite}(\Phi,\mathcal{C}) &\to \mathbf{Suite}(\Phi,\mathcal{C}) \\ X = (X_n)_{n\in\mathbb{N}} &\mapsto s_-(X) = ((s_-(X))_n)_{n\in\mathbb{N}} \end{aligned}$$

  where $(s_-(X))_n = X_{n+1}$ equipped with the Φ_n-action by the restriction through $\phi_{n,1}(-,1) : \Phi_n \to \Phi_{n+1}$.

- Suppose that $\mathcal{C}$ admits relevant colimits. The functor s_- admits a left adjoint

$$\begin{aligned} s_+ : \mathbf{Suite}(\Phi,\mathcal{C}) &\to \mathbf{Suite}(\Phi,\mathcal{C}) \\ X = (X_n)_{n\in\mathbb{N}} &\mapsto s_+(X) = ((s_+(X))_n)_{n\in\mathbb{N}} \end{aligned}$$

  where
$$(s_+(X))_n = \begin{cases} \emptyset & n = 0 \\ \mathrm{Ind}_{\Phi_{n-1}\times 1}^{\Phi_n} X_{n-1} & n \geqq 1 \end{cases}$$

- Suppose that $\mathcal{C}$ admits relevant colimits. Then the above adjunction

$$(s_+, s_-) : \mathbf{Suite}(\Phi,\mathcal{C}) \to \mathbf{Suite}(\Phi,\mathcal{C})$$

  extends naturally to an adjunction

$$(s_+, s_-) : \mathbf{Spect}_F^{\Phi}(\mathcal{C}) \to \mathbf{Spect}_F^{\Phi}(\mathcal{C})$$

  such that the following squares commute:

$$\begin{tikzcd} \mathbf{Spect}_F^{\Phi}(\mathcal{C}) \arrow[r, "s_-"] \arrow[d, "\prod_n \mathrm{Ev}_n"'] & \mathbf{Spect}_F^{\Phi}(\mathcal{C}) \arrow[d, "\prod_n \mathrm{Ev}_n"] \\ \mathbf{Suite}(\Phi,\mathcal{C}) \arrow[r, "s_-"] & \mathbf{Suite}(\Phi,\mathcal{C}) \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Spect}_F^{\Phi}(\mathcal{C}) \arrow[r, "s_+"] \arrow[d, "\prod_n \mathrm{Ev}_n"'] & \mathbf{Spect}_F^{\Phi}(\mathcal{C}) \arrow[d, "\prod_n \mathrm{Ev}_n"] \\ \mathbf{Suite}(\Phi,\mathcal{C}) \arrow[r, "s_+"] & \mathbf{Suite}(\Phi,\mathcal{C}) \end{tikzcd}$$

[Ayoub, Def,4.3.16, p.479]

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories equipped with each Φ-symmetric endofunctor F and F'.

- Let $K : \mathcal{C} \to \mathcal{C}'$ be a functor and $\tau : F' \circ K \to K \circ F$ a natural transformation. We sau that τ is <u>Φ-symmetric</u> when the composite:

$$F'^{\circ n} \circ K \xrightarrow{\tau} F'^{\circ(n-1)} \circ K \circ F \xrightarrow{\tau} \cdots \xrightarrow{\tau} K \circ F^{\circ n}$$

  is Φ_n-equivariant.

- If τ is symmetric, the <u>extension</u> of K (following τ) is the functor

$$\begin{aligned} K_\tau : \mathbf{Spect}_F^\Phi(\mathcal{C}) &\to \mathbf{Spect}_{F'}^\Phi(\mathcal{C}') \\ \mathbf{X} &\mapsto K_\tau(\mathbf{X}) = ((K_\tau(\mathbf{X}))_n = K(\mathbf{X}_n))_{n \in \mathbb{N}} \end{aligned}$$

  where the assembly map of $K_\tau(\mathbf{X})$ is given by the composite:

$$F'(K(\mathbf{X}_n)) \xrightarrow{\tau} K(F(\mathbf{X}_n)) \to K(\mathbf{X}_{n+1})$$

[Ayoub, Prop,4.3.19, p.480]

Let α be an upper bound cardinal of the cardinal of the monoid Φ.

- Suppose that $\mathcal{C}$ is α-presentable in the sennse of Definition 4.2.16 and that the functor
- F admits a right adjoint G which is α-accessible.

Then the category $\mathbf{Spect}_F^\Phi(\mathcal{C})$ is equally α-presentable.

We now study the preceeding general theory of symmetric spectra in the model category setting. For this purpose, we make the following hypothesis:

Hypothesis, 4.3.2, p.481

Given a Quillen adjunction

$$(F, G) : \mathfrak{M} = (\mathfrak{M}, \mathbf{W}, \mathbf{Cof}, \mathbf{Fib}) \circlearrowleft$$

with F : Φ-symmetric in the sense of Definition 4.3.4.

[Ayoub, Def,4.3.20, p.481]

Let $f : \mathbf{X} \to \mathbf{Y}$ be an arrow of $\mathbf{Spect}_F^{\Phi}(\mathfrak{M})$.

- We say that f is a levelwise weak equivalence (resp. levelwise cofibration,levelwise fibration) if for any $n \in \mathbb{N}$ the arrow $f_n : \mathbf{X}_n \to \mathbf{Y}_n$ is a weak equivalence (resp. cofibration, fibration).
- Denote by $\mathbf{W}_{niv}$ (resp. $\mathbf{Cof}_{niv}, \mathbf{Fib}_{niv}$) the class of levelwise weak equivalences (resp. levelwise cifibrations, levelwise fibrations).
- We say f is a projective cofibration (resp. iinjective fibration) when it posseses a left (resp. right) lifting property with respect to levelwise trivial fibrations (resp. levelwise trivial cofibrations).
- Denote by $\mathbf{Cof}_{proj}$ (resp. $\mathbf{Fib}_{inj}$ the class of projective cofibrations (resp. injective fibrations).

[Ayoub, Prop,4.3.21, p.481; Def,4.3.22, p.483]

Suppose that the model category $\mathfrak{M}$ is presentable by cofibrations (see Definition 4.2.39) and that the functor G is accessible.
Suppose also that for $(m, n) \in \mathbb{N}^2$, the group homomorphisms $\phi_{m,n}(-, 1) : \Phi_m \to \Phi_{m+n}$ are injective.
Then the following two quadruplets are model categories, which are again presentable by cofibrations:

$$\left(\mathbf{Spect}_F^{\Phi}(\mathfrak{M}), \mathbf{W}_{niv}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{niv}\right), \ \underline{\text{unstable projective model structure}}$$

$$\left(\mathbf{Spect}_F^{\Phi}(\mathfrak{M}), \mathbf{W}_{niv}, \mathbf{Cof}_{niv}, \mathbf{Fib}_{inj}\right), \ \underline{\text{unstable injective model structure}}$$

[Ayoub, Lem,4.3.24, p.483]

For $p \in \mathbb{N}$, the adjunction

$$\left(\mathrm{Sus}^p_{F,\Phi}, \mathrm{Ev}_p\right) : \mathbf{Spect}_F^{\Phi}(\mathfrak{M}) \to \mathfrak{M}$$

is a Quillen adjunction with respect to the unstable projective structure of $\mathbf{Spect}_F^{\Phi}(\mathfrak{M})$.

[Ayoub, w^p_X, p.484]

Recall the adjunction in Def.4.3.10, p.476:

$$\left(\mathrm{Sus}^p_{F,\Phi}, \mathrm{Ev}_p\right) : \mathcal{C} \underset{\mathrm{Oub}_1^{\Phi_p}}{\overset{\mathrm{Ind}_1^{\Phi_p}}{\rightleftarrows}} \mathbf{Rep}(\Phi_p, \mathcal{C}) \underset{\underline{\mathrm{Ev}}_p}{\overset{\underline{\mathrm{Sus}}^p_{F,\Phi}}{\rightleftarrows}} \mathbf{Spect}^{\Phi}_F(\mathcal{C})$$

Notice

$$\mathrm{Ev}_{p+1}\left[\mathrm{Sus}^p_{F,\Phi}(X)\right] = \mathrm{Oub}_1^{\Phi_{p+1}}\left(\mathrm{Ind}_{\Phi_1}^{\Phi_{p+1}} F(X)\right)$$

In fact, <u>as F commutes with colimits</u> ,

$$\begin{aligned}
\mathrm{Ev}_{p+1}\left[\mathrm{Sus}^p_{F,\Phi}(X)\right] &= \left[\mathrm{Sus}^p_{F,\Phi}(X)\right]_{p+1} \\
&= \left[\underline{\mathrm{Sus}}^p_{F,\Phi}\left(\mathrm{Ind}_1^{\Phi_p} X\right)\right]_{p+1} \\
&\overset{Def.4.3.9,p.476}{=} \mathrm{Oub}_1^{\Phi_{p+1}}\left(\mathrm{Ind}_{\Phi_1\times\Phi_p}^{\Phi_{p+1}}\left(F(\mathrm{Ind}_1^{\Phi_p} X)\right)\right) \qquad (4) \\
&\overset{F\circ\mathrm{colim}\equiv\mathrm{colim}\circ F}{\equiv} \mathrm{Oub}_1^{\Phi_{p+1}}\left(\mathrm{Ind}_{\Phi_1\times\Phi_p}^{\Phi_{p+1}}\left(\mathrm{Ind}_{\Phi_1\times 1}^{\Phi_1\times\Phi_p}(F(X))\right)\right) \\
&= \mathrm{Oub}_1^{\Phi_{p+1}}\left(\mathrm{Ind}_{\Phi_1}^{\Phi_{p+1}} F(X)\right)
\end{aligned}$$

On the other hand, applying an obvious nonequivariaant map for an $(G \supseteqq)H$-set Y: $(Y \xrightarrow{\cong} H \times_H Y \subseteqq G \times_H U = \mathrm{Oub}_1^G \mathrm{Ind}_H^G Y)$, for our $(\Phi_{p+1} \supseteqq)\Phi_1$-set $F(X)$, we get

$$F(X) \to \mathrm{Oub}_1^{\Phi_{p+1}}\left(\mathrm{Ind}_{\Phi_1}^{\Phi_{p+1}} F(X)\right) \qquad (5)$$

By composing (5)with(4), we obtain

$$F(X) \to \mathrm{Ev}_{p+1}\left[\mathrm{Sus}^p_{F,\Phi}(X)\right]$$

whose adjoint is nothing but our desired

$$w^p_X : \mathrm{Sus}^{p+1}_{F,\Phi}(F(X)) \to \mathrm{Sus}^p_{F,\Phi}(X)$$

[Ayoub, $\mathcal{R}, \mathcal{R}_\beta$, Def,4.3.29, p.484]

- Denote by $\mathcal{R}$ (resp. $\mathcal{R}_\beta$) the class of arrows w_X^p with $p \in \mathbb{N}$ and X a cofibrant (resp. and β-accesible) object of $\mathfrak{M}$.
- Denote by $\mathbf{W}_{\mathcal{R}}$ (resp. $\mathbf{W}_{\mathcal{R}_\beta}$) the class of $\mathcal{R}$-weak equivalences (resp. $\mathcal{R}_\beta$-weak equivalences) in the sense of Def.4.2.64.

Suppose that $\mathfrak{M}$ is left proper, presentable by cofibrations, and that G is accessible.

- Then the stable projective (resp. stable injective model structure on $\mathbf{Spect}_F^\Phi(\mathfrak{M})$ is the Bousfield localisation of the unstable projective (resp. unstable injective) model structure of Def.4.3.22 with respect to the class $\mathcal{R}$.
- Denote by $\mathbf{W}_{st}$ the class of the $\mathcal{R}$-weak equivalences, which shall call stable equivalences.
- We also denote $\mathbf{Fib}_{proj-st}$ (resp. $\mathbf{Fib}_{inj-st}$) the class of $\mathcal{R}$-projective fibrations (resp. $\mathcal{R}$-injective fibrations), which we shall call stable projective fibrations (resp. stable injective fibrations).
- Finally, we set

$$\mathbf{Ho}_{st}\left(\mathbf{Spect}_F^\Phi(\mathfrak{M})\right) = \mathbf{Spect}_F^\Phi(\mathfrak{M})\left[\mathbf{W}_{st}^{-1}\right]$$

[Ayoub, Prop,4.3.30, p.485]

A levelwise fibrant object $\mathbf{X}$ of $\mathbf{Spect}_F^\Phi(\mathfrak{M})$ is $\mathcal{R}$-local if and only if for any $n \in \mathbb{N}$ the morphism obtained by adjunction of the assembly morphism

$$\mathbf{X}_n \to G\mathbf{X}_{n+1}$$

is a weak equivalence.

[Ayoub, Def,4.3.31, p.486]

A (F, Φ)-spectrum $\mathbf{X}$ is called a $\Omega_{F,\Phi}$-spectrum (or simply Ω-spectrum) if for any $n \in \mathbb{N}$, the arrow

$$\mathbf{X}_n \to (\mathsf{R}G)\mathbf{X}_{n+1}$$

is invertible in $\mathbf{Ho}(\mathfrak{M})$.

[Ayoub, Lem,4.3.34, p.486]

Suppose two model categories, which are presentable by cofibrations, as well as Φ-symmetric endofunctors F and F' which are left Quillen functors having accessible right adjoints. Let

$$K : \mathfrak{M} \to \mathfrak{M}'$$

be a left Quillen functor equipped with an invertible Φ-symmetric natural transformation:

$$\tau : F' \circ K \xrightarrow{\sim} K \circ F \qquad \left(\begin{tikzcd} \mathfrak{M} \arrow[r, "K"] \arrow[d, "F"'] \arrow[dr, phantom, "\not\swarrow\simeq"] & \mathfrak{M}' \arrow[d, "F'"] \\ \mathfrak{M} \arrow[r, "K"'] & \mathfrak{M}' \end{tikzcd} \right)$$

Then, the functor:

$$K_\tau : \mathbf{Spect}^{\Phi}_{F}(\mathfrak{M}) \to \mathbf{Spect}^{\Phi}_{F'}(\mathfrak{M}')$$

is a left Quillen functor with respect to the stable projective structures.

[Ayoub, Prop,4.3.35, p.486]

If (F, G) is a Quillen equivalence, then the adjunction

$$(\mathrm{Sus}^{0}_{F,\Phi}, \mathrm{Ev}_0) : \mathfrak{M} \to \mathbf{Spect}^{\Phi}_{F}(\mathfrak{M})$$

is a Quillen equivalence when $\mathbf{Spect}^{\Phi}_{F}(\mathfrak{M})$ is equipped with the stable projective structure.

[Ayoub, Def,4.3.36, p.487]

For $\tau \in \Phi_2$, denote by $\tau_{(n)} \in \Phi_{n+1}$ the element $\tau_1 \cdots \tau_n$ with $\tau_i = \phi_{i-1,2,n-i}(1, \tau, 1)$ the image of τ by the morphism

$$\begin{aligned} \phi_{i-1,2,n-i}(1, -, 1) : \Phi_2 &= 1 \times \Phi_2 \times 1 \\ &\to \Phi_{i-1} \times \Phi_2^{\times} \Phi_{n-i} \to \Phi_{n+1} \end{aligned}$$

Then the element $\tau \in \Phi_2$ is called symmetric when for any $g \in \Phi_n$ we have:

$$\tau_{(n)} \cdot \phi_{n,1}(g, 1) = \phi_{1,n}(1, g) \cdot \tau_{(n)}$$

where $\phi_{n,m} : \Phi_m \times \Phi_n \to \Phi_{m+n}$ determines the monoidal structure of Φ.

[Ayoub, Ex.4.3.37 p.487]

The transposition $\tau = (12) \in \Sigma_2$ is a symmetric element of the monoid of symmetric groups Σ of Example 4.3.1. In effect, the permutation $\tau_{(n)}$ is equal to the product of transpositions:

$$(12)(23)\cdots(n-1,n)(n,n+1) = (1,2,3,\cdots,n,n+1)$$

In particular, its restriction to $\{1,\ldots,n\}$ is given by $i \rightsquigarrow i+1$. It follows that for $g \in \Sigma_n$, we have

$$\tau_{(n)}(g \circ 1) = (1 \bullet g)\tau_{(n)}$$

[Ayoub, Th.4.3.38 p.488]

If $\exists \tau \in \Psi_2$, a symmetric element. Then w.r.t. the model category $\left(\mathbf{Spect}_F^{\Psi}(\mathfrak{M}), \mathbf{W}_{st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-st}\right)$,

1. The adjunction
$$(s_+, s_-) : \mathbf{Spect}_F^{\Psi}(\mathfrak{M}) \rightleftarrows \mathbf{Spect}_F^{\Psi}(\mathfrak{M})$$
is a Quillen equivalence.

2. The adjunction
$$(F_\tau, G_{\tau'}) : \mathbf{Spect}_F^{\Psi}(\mathfrak{M}) \rightleftarrows \mathbf{Spect}_F^{\Psi}(\mathfrak{M})$$
is a Quillen equivalence.

3. The natural transformation $t_\tau : F_\tau \to s_-$ induces an isomorphism
$$\mathsf{L}F_\tau \xrightarrow{\sim} \mathsf{R}s_-$$

[Ayoub, Def,4.3.39, p.489]

Given a morphism of graded monoids $\alpha : \Phi \to \Phi'$. An element $\sigma \in \Phi'_2$ is called <u>symmetric relative to Φ</u> when for $n \in \mathbb{N}$ and $g \in \Phi_n$ the following relation is satisfied:

$$\sigma_{(n)}\phi'_{n,1}(\alpha(g), 1) = \phi'_{1,n}(1, \alpha(g)) \cdot \sigma_{(n)}.$$

with $\sigma_{(n)} = \sigma_1 \cdots \sigma_n$ for $\sigma_i = \phi_{i-1,2,n-i}(1, \sigma, 1)$.

<u>The goal of this paragraph is to prove the following:</u>

[Ayoub, Th.4.3.40 p.489]

Suppose we are given

- $\begin{cases} F \\ F' \end{cases} : \begin{cases} \Psi \\ \Psi' \end{cases}$ -summetric endofunctor of $\mathfrak{M}$, which are respectively left Quillen functor admitting accesible right adjoint;
- a graded monoid morphism $\alpha : \Psi \to \Psi'$;
- a Ψ-equivariant natural transformation $F \to F'$,

which satisfy the following conditions:

1. $\exists \tau' \in \Psi'_2$, a symmetric element, s.t. $\sigma = (\tau')^{-1}$ is symmetric w.r.t. Ψ;
2. for any cofibrant object $X \in \mathfrak{M}$, $F(X) \to F'(X)$ is a weak equivalence;
3. the functor $F'_\sigma : \mathbf{Spect}^{\Psi}_{F}(\mathfrak{M}) \to \mathbf{Spect}^{\Psi'}_{F'}(\mathfrak{M})$ is a left Quillen equivalence,

then the adjunction

$$\left((F', \Psi') \otimes_{F,\Psi} -, \mathrm{Oub}^{F',\Psi'}_{F,\Psi}\right) : \mathbf{Spect}^{\Psi}_{F}(\mathfrak{M}) \rightleftarrows \mathbf{Spect}^{\Psi'}_{F'}(\mathfrak{M})$$

is a Quillen equivalence w.r.t. the stable projective model structures.

[Ayoub, Prop,4.3.42, p.489]

- Let $F \to F'$ be a Φ-equivariant natural transformation bwtween Φ-symmetric left Quillen endofunctors of $\mathfrak{M}$ which admit accessible right adjoints.
- Suppose that the arrow $F(X) \to F'(X)$ is a weak equivalence for cofibrant $X \in \mathrm{Ob}(\mathfrak{M})$.

Then the adjunction

$$\left((F', \Psi') \otimes_{F,\Psi} -, \mathrm{Oub}^{F',\Psi'}_{F,\Psi}\right) : \mathbf{Spect}^{\Psi}_{F}(\mathfrak{M}) \rightleftarrows \mathbf{Spect}^{\Psi'}_{F'}(\mathfrak{M})$$

is a Quillen equivalence w.r.t. the levelwise projective model structures.

[Ayoub, Lem.4.3.59 p.498]

Let $f : \mathbf{X} \to \mathbf{Y}$ be a morphism of F-spectra. Suppose that $f_n : \mathbf{X}_n \to \mathbf{Y}_n$ is a weak equivalence of $\mathfrak{M}$ for $n \geq N$. Then f is a stable equivalence.

[Ayoub, Th.4.3.61 p.499]

For a levelwise **fibrant** F-spectrum $\mathbf{X}$, the homotopy colimit $\mathbb{N}$-sequence

$$\mathbf{X} \xrightarrow{\lambda_{\mathbf{X}}} \Lambda(\mathbf{X}) \to \cdots \to \Lambda^{\circ n} \xrightarrow{\lambda_{\Lambda^{\circ n}(\mathbf{X})}} \Lambda^{\circ (n+1)} \to \cdots$$

is an Ω_F-spectrum. Furthermore, the evident morphism

$$\mathbf{X} \to \mathrm{L}\,\mathrm{Colim}_{n \in \mathbb{N}} \Lambda^{\circ n}\mathbf{X}$$

is a stable equivalence.

The monoidal structure on $\mathbf{Suite}(\Sigma, \mathcal{C})$ [Ayoub, Def,4.3.63, p.500]

Consider a unital monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ such that $\mathcal{C}$ admits coproducts and that, for any $A \in \mathrm{Ob}(\mathcal{C})$, the following functors commute:

$$A \otimes -, \quad \text{and} \quad - \otimes A$$

Then, $\mathbf{Suite}(\Sigma, \mathcal{C})$ (defined in Def.4.3.3, Ex.4.3.1) is also provided with a unital monoidal structure as follows:

- Let $X = (X_n)_{n \in \mathbb{N}}$ and $Y = (Y_n)_{n \in \mathbb{N}}$ be two symmetric sequences in $\mathcal{C}$. We define a new symmetric sequence $X \otimes Y$ by
$$(X \otimes Y)_n \doteq \coprod_{i+j=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X_i \otimes Y_j)$$
- The monoidal unit object is provided by $\mathbf{1}_\Sigma = (\mathbf{1}, \emptyset, \ldots, \emptyset, \ldots)$.
- The associativity isomorphism of the above monodal product $\otimes$ is prrovided at the level n by the composite:
$$\begin{aligned}(X \otimes (Y \otimes Z))_n &\simeq \coprod_{i+j+k=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j \times \Sigma_k}^{\Sigma_n} (X_i \otimes (Y_j \otimes Z_k)) \\ &\simeq \coprod_{i+j+k=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j \times \Sigma_k}^{\Sigma_n} ((X_i \otimes Y_j) \otimes Z_k) \simeq ((X \otimes Y) \otimes Z)_n\end{aligned}$$

The closed structure of $\mathbf{Suite}(\Sigma, \mathcal{C})$ [Ayoub, Lem.4.3.64, p.500]

Suppose $\mathcal{C}$ is left (resp. right) closed and admits relevant limits. Then the monoidal category $\mathbf{Suite}(\Sigma, \mathcal{C})$ is left (resp. right) closed.

Proof.

- For $X = (X_n)_{n\in\mathbb{N}}, Z = (Z_n)_{n\in\mathbb{N}} \in \mathbf{Suite}(\Sigma, \mathcal{C})$ and $(i, n) \in \mathbb{N}^2$, denote by

$$\underline{\mathrm{Hom}}_g^{\Sigma_i}(X_i, Z_{i+n})$$

  the sub-object of Σ_i-invariants of $\underline{\mathrm{Hom}}_g(X_i, Z_{i+n})$ for the action:

$$t \in \Sigma_i \rightsquigarrow \underline{\mathrm{Hom}}_g(X_i, Z_{i+n}) \xrightarrow{\underline{\mathrm{Hom}}_g(t^{-1}, Z_{i+n})} \underline{\mathrm{Hom}}_g(X_i, Z_{i+n}) \xrightarrow{\underline{\mathrm{Hom}}_g(X_i, \phi_{i,n}(t,1))} \underline{\mathrm{Hom}}_g(X_i, Z_{i+n})$$

- Σ_n acts on $\underline{\mathrm{Hom}}_g(X_i, Z_{i+n})$ by $\underline{\mathrm{Hom}}_g(X_i, \phi_{1,n}(1, -))$. This Σ_n action commutes with the Σ_i action above, and so restricts to $\underline{\mathrm{Hom}}_g^{\Sigma_i}(X_i, Z_{i+n})$.
- Now we can define a Σ-symmetric sequence $\underline{\mathrm{Hom}}_g(X, Z)$ by:

$$\underline{\mathrm{Hom}}_g(X, Z)_n = \prod_{i\in\mathbb{N}} \underline{\mathrm{Hom}}_g^{\Sigma_i}(X_i, Z_{i+n}).$$

- This $\underline{\mathrm{Hom}}_g(X, Z)$ actually represents $\mathrm{hom}_{\mathbf{Suite}(\Sigma,\mathcal{C})}(X \otimes -, Z)$. This is because, for $Y = (Y_n)_{n\in\mathbb{N}}$,

$$\begin{aligned}
&\mathrm{hom}_{\mathbf{Suite}(\Sigma,\mathcal{C})}\left(Y, \underline{\mathrm{Hom}}_g(X, Z)\right) = \prod_{n\in\mathbb{N}} \mathrm{hom}_{\mathcal{C}}^{\Sigma_n}\left(Y_n, \underline{\mathrm{Hom}}_g(X, Z)_n\right) \\
&= \prod_{n\in\mathbb{N}}\prod_{i\in\mathbb{N}} \mathrm{hom}_{\mathcal{C}}^{\Sigma_n}\left(Y_n, \underline{\mathrm{Hom}}_g^{\Sigma_i}(X_i, Z_{i+n})\right) = \prod_{(n,i)\in\mathbb{N}^2} \mathrm{hom}_{\mathcal{C}}^{\Sigma_i\times\Sigma_n}(X_i \otimes Y_n, \mathrm{Res}^{\Sigma_{i+n}}_{\Sigma_i\times\Sigma_n} Z_{i+n}) \\
&= \prod_{(n,i)\in\mathbb{N}^2} \mathrm{hom}_{\mathcal{C}}^{\Sigma_{i+n}}\left(\mathrm{Ind}^{\Sigma_{i+n}}_{\Sigma_i\times\Sigma_n} X_i \otimes Y_n, Z_{i+n}\right) = \prod_{m\in\mathbb{N}} \mathrm{hom}_{\mathcal{C}}^{\Sigma_m}\left(\coprod_{i+n=m} \mathrm{Ind}^{\Sigma_{i+n}}_{\Sigma_i\otimes\Sigma_n} X_i \otimes Y_n, Z_m\right) \\
&= \prod_{(n,i)\in\mathbb{N}^2} \mathrm{hom}_{\mathcal{C}}^{\Sigma_{i+n}}((X \times Y)_m, Z_m) = \mathrm{hom}_{\mathbf{Suite}(\Sigma,\mathcal{C})}\left(X \otimes Y, Z\right).
\end{aligned}$$

- The other case follows from this case by $\otimes$-duality.

□

The symmetric monoidal structure of **Suite**$(\Sigma, \mathcal{C})$ - 1 [Ayoub, p.500; Lem.4.3.65, p.500; Lem.4.3.66, p.501]

- For $n = i + j$, define $\theta_{i,j} \in \Sigma_n$ by:

$$\theta_{i,j}(a) = \begin{cases} a+j & \text{if} \quad 1 \le a \le i \\ a-i & \text{if } i+1 \le a \le i+j \end{cases}$$

Then $\theta_{j,i} = \theta_{i,j}^{-1}$ and the following diagram is commutative:

$$\begin{tikzcd} \Sigma_i \times \Sigma_j \arrow[r, "\phi_{i,j}"] \arrow[d, "\tau\ \text{(the permutation of factors)}"'] & \Sigma_n \arrow[d, "\theta_{i,j}(-)\theta_{i,j}^{-1}"] \\ \Sigma_j \times \Sigma_i \arrow[r, "\phi_{j,i}"] & \Sigma_n \end{tikzcd}$$

Proof. For $(A_i, B_j) \in \Sigma_i \times \Sigma_j$ $(\subset GL_i \times GL_j)$,

$$\begin{aligned} \theta_{i,j}\left(\phi_{i,j}(A_i, b_j)\right)\theta_{i,j}^{-1} &= \begin{pmatrix} 0 & I_j \\ I_i & 0 \end{pmatrix}\begin{pmatrix} A_i & 0 \\ 0 & B_j \end{pmatrix}\begin{pmatrix} 0 & I_i \\ I_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_j \\ A_i & 0 \end{pmatrix}\begin{pmatrix} 0 & I_i \\ I_j & 0 \end{pmatrix} \\ &= \begin{pmatrix} B_j & 0 \\ 0 & A_i \end{pmatrix} = \phi_{j,i}\tau(A_i, B_j). \end{aligned}$$

□

- For $X = (X_n)_{n\in\mathbb{N}}, Y = (Y_n)_{n\in\mathbb{N}} \in$ **Suite**$(\Sigma, \mathcal{C})$, the composite:

$$X_i \otimes Y_j \xrightarrow{\tau} Y_j \otimes X_i \to \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i) \xrightarrow{\theta_{j,i}\cdot} \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i)$$

is $\Sigma_i \times \Sigma_j$-equivariant for:

- the product action on $X_i \otimes Y_j$.
- the aaction obtained by restricting via $\phi_{i,j} : \Sigma_i \times \Sigma_j \to \Sigma_n$ of the action of Σ_n on $\mathrm{Ind}_{\Sigma_i\times\Sigma_j}^{\Sigma_n}(Y_j \otimes X_i)$.

Proof. For $(u, v) \in \Phi_i \times \Phi_j$, we have the commutative diagram:

$$\begin{tikzcd} X_i \otimes Y_j \arrow[r, "\tau"] \arrow[d, "(u,v)"'] & Y_j \otimes X_i \arrow[r] \arrow[d, "(v,u)"'] & \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i) \arrow[r, "\theta_{j,i}\cdot"] \arrow[d, "\phi_{j,i}(v,u)"] & \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i) \arrow[d, "\theta_{j,i}\phi_{j,i}(v,u)\theta_{j,i}^{-1}"] \\ X_i \otimes Y_j \arrow[r, "\tau"] & Y_j \otimes X_i \arrow[r] & \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i) \arrow[r, "\theta_{j,i}\cdot"] & \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i) \end{tikzcd}$$

Here, by the above result, $\theta_{j,i}\phi_{j,i}(v, u)\theta_{j,i}^{-1} = \phi_{i,j}(u, v)$, so the claim follows. □

- By the above result, the composite:

$$X_i \otimes Y_j \xrightarrow{\tau} Y_j \otimes X_i \to \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i) \xrightarrow{\theta_{j,i}\cdot} \mathrm{Res}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}\mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i)$$

is $\Sigma_i \times \Sigma_j$-equivariant, and so we get the following Σ_n-equivariant morphism:

$$\theta : \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(X_i \otimes Y_j) \to \mathrm{Ind}_{\Sigma_j\times\Sigma_i}^{\Sigma_n}(Y_j \otimes X_i),$$

and so, by passage to the coprod $\coprod_{i+j=n}$, we get a natural morphism of symmetric sequences:

$$\theta : X \otimes Y \to Y \otimes X.$$

We shall quickly verify that θ makes the monoidal structure on **Suite**$(\Sigma, \mathfrak{M})$ symmetric:

The symmetric monoidal structure of $\mathbf{Suite}(\Sigma, \mathcal{C})$ - 2 [Ayoub, Lem.4.3.66, p.501; Prop.4.3.67, p.502]

- The arrow $\theta : X \otimes Y \to Y \otimes X$ is involutive, i.e. $\theta^2 = \mathrm{id}_{X \otimes Y}$. Furhermore,the following diagram is commutative:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\sim"] \arrow[d, "\sim"'] & Z \otimes (X \otimes Y) \arrow[r, "\sim"] & Z \otimes (Y \otimes X) \arrow[d, "\sim"] \\
X \otimes (Y \times Z) \arrow[r, "\sim"] & (Y \otimes Z) \otimes X \arrow[r, "\sim"] & (Z \otimes Y) \otimes X
\end{tikzcd}$$

Proof. For this purpose, given a symmetric sequence $T = (T_n)_{n \in \mathbb{N}}$ and evaluate the functor $\hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}(-, T)$ on θ^2 and on the displayed diagram:

- The map

$$\begin{cases} \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}(\theta, T) : \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}(Y \otimes X, T) & \to \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}(X \otimes Y, T) \\ \prod_{j+i=n} \hom_{\mathbf{Rep}(\Sigma_j \times \Sigma_i, \mathcal{C})}(Y_j \otimes X_i, T) & \to \prod_{i+j=n} \hom_{\mathbf{Rep}(\Sigma_i \times \Sigma_j, \mathcal{C})}(X_i \otimes Y_j, T) \\ (\gamma_{j,i})_{j+i=n} & \mapsto \left(X_i \otimes Y_j \xrightarrow{\tau} Y_j \otimes X_i \xrightarrow{\gamma_{j,i}} T_n \xrightarrow{\theta_{j,i}} T_n \right)_{i+j=n} \end{cases}$$

  Since $\tau^2 = \mathrm{id}$ and $\theta_{ji}\theta_{ij} = 1$, we see immediately that $\hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}(\theta^2, T)$ is the identity.

- To show the commutativity of the diagram, compute the two composites seperately:

$$\begin{cases} \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}((Z \otimes Y) \otimes X, T) \to \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}((Y \otimes Z) \otimes X, T) \to \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}(X \otimes (Y \otimes Z), T) \\ \prod_{k+j+i=n} \hom_{\mathbf{Rep}(\Sigma_k \times \Sigma_j \times \Sigma_i, \mathcal{C})}((Z_k \otimes Y_j) \otimes X_i, T) \to \prod_{i+j+k=n} \hom_{\mathbf{Rep}(\Sigma_i \times \Sigma_j \times \Sigma_k, \mathcal{C})}(X_i \otimes (Y_j \otimes Z_k), T) \\ (\gamma_{k,j,i})_{k+j+i=n} \mapsto \left(X_i \otimes (Y_j \otimes Z_k) \xrightarrow{\tau} (Y_j \otimes Z_k) \otimes X_i \xrightarrow{\tau} (Z_k \otimes Y_j) \otimes X_i \xrightarrow{\gamma_{k,j,i}} T \xrightarrow{\phi_{j+k,i}(\theta_{j,k},1)} T \xrightarrow{\theta_{j+k,i}} T \right)_{i+j+k=n} \end{cases}$$

$$\begin{cases} \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}(Z \otimes (Y \otimes X), T) \to \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}((Y \otimes X) \otimes Z, T) \to \hom_{\mathbf{Suite}(\Sigma,\mathcal{C})}((X \otimes Y) \otimes Z, T). \\ \prod_{k+j+i=n} \hom_{\mathbf{Rep}(\Sigma_k \times \Sigma_j \times \Sigma_i, \mathcal{C})}(Z_k \otimes (Y_j \otimes X_i), T) \to \prod_{i+j+k=n} \hom_{\mathbf{Rep}(\Sigma_i \times \Sigma_j \times \Sigma_k, \mathcal{C})}((X_i \otimes Y_j) \otimes Z_k, T) \\ (\gamma_{k,j,i})_{k+j+i=n} \mapsto \left((X_i \otimes Y_j) \otimes Z_k \xrightarrow{\tau} Z_k \otimes (X_i \otimes Y_j) \xrightarrow{\tau} Z_k \otimes (Y_j \otimes X_i) \xrightarrow{\gamma_{k,j,i}} T \xrightarrow{\phi_{k,i+j}(1,\theta_{j,i})} T \xrightarrow{\theta_{k,i+j}} T \right)_{i+j+k=n} \end{cases}$$

  The commutation of the diagram then follows from the commutation of the corresponding diagram in the symmetric monoidal category $\mathcal{C}$ as well as the relation $\theta_{j+k,i}\phi_{j+k,i}(\theta_{j,k}, 1) = \theta_{k,i+j}\phi_{k,i+j}(1, \theta_{j,i})$ □

From these discussions, we now have the following:

[Ayoub, Prop.4.3.67, p.502]

Let $(\mathcal{C}, \otimes)$ be a unital symmetric monoidal closed category. Suppose $\mathcal{C}$ admits relevant colimits and relevant limits. Then $(\mathbf{Suite}(\Sigma, \mathcal{C}), \otimes)$ is also a unital symmetric monoidal closed cateogry.

[Ayoub, Def,4.3.68, Prop.4.3.69, p.502]

Let T be an object of $\mathcal{C}$.

- Denote by $\mathcal{S}^T$ the symmetric sequence given at the level $n \in \mathbb{N}$ by the object $\mathcal{S}^T_n = T^{\otimes n}$ equipped with the Σ_n action induced by the permutation of factors.
- Denote by m the morphism ofsymmetric sequences $\mathcal{S}^T \otimes \mathcal{S}^T \to \mathcal{S}^T$ given at the level n by the coproduct decompositions $n = i + j$ with evident arrows:

$$\mathrm{Ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \left(T^{\otimes i} \otimes T^{\otimes j} \right) \to T^{\otimes n}$$

Then the sequence $\mathcal{S}^T$, equipped wih the coupling $m : \mathcal{S}^T \otimes \mathcal{S}^T \to \mathcal{S}^T$ and the unit $\mathbf{1}_\Sigma \to \mathcal{S}^T$, is a unital commutative algebra in the monoidal category in the mnoidal category $(\mathbf{Suite}(\Sigma, \mathcal{C}), \otimes)$ given in Definition 4.3.63.

The unital symmetric monoidal product $-\otimes-$ on $\mathbf{Spect}^{\Sigma}_T(\mathcal{C})$ - 1 [Ayoub, p.503; Prop.4.3.70, p.503]

- Denote by $\mathbf{Mod}_g(S^T)$ (resp. $\mathbf{Mod}_d(S^T)$ the caetgory of left (resp. right) S^T-modules in the mnoidal category $(\mathbf{Suite}(\Sigma,\mathcal{C}),\otimes)$ given in Definition 4.3.63.
- Since S^T is a commutative algebra, there is an isomorphism:

$$\mathbf{Mod}_g(S^T)\simeq\mathbf{Mod}_d(S^T)$$
$$(S^T\otimes M\to M)\mapsto(M\otimes S^T\simeq S^T\otimes M\to M)$$

- The category $\mathbf{Spect}^{\Sigma}_T(\mathcal{C})$ is tautologically isomorphic to the caegory $\mathbf{Mod}_g(S^T)$ of left S^T-modules in $\mathbf{Suite}(\Sigma,\mathcal{C})$.

Proof. The inverse isomorphisms are provided as follows:

$\underline{\mathbf{Spect}^{\Sigma}_T(\mathcal{C})\to\mathbf{Mod}_g(S^T)}$ Given $\mathbf{X}\in\mathbf{Spect}^{\Sigma}_T(\mathcal{C})$, take

$$(\mathbf{X}_n)_{n\in\mathbb{N}}\in\mathbf{Suite}(\Sigma,\mathcal{C})\leftarrow\mathbf{Mod}_g(S^T),$$

which is actually in $\mathbf{Mod}_g(S^T)$ by the coupling:

$$S^T\otimes(\mathbf{X}_n)_{n\in\mathbb{N}}\to(\mathbf{X}_n)_{n\in\mathbb{N}}$$

given on degree n by:

$$\begin{tikzcd}
\left(S^T\otimes(\mathbf{X}_m)_{m\in\mathbb{N}}\right)_n \arrow[r] \arrow[d, equal, "\text{Def.4.3.63; Def.4.3.68}"'] & \mathbf{X}_n \arrow[d, equal] \\
\coprod_{i+j=n}\mathrm{Ind}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}(T^{\otimes i}\otimes\mathbf{X}_j) \arrow[r, "\coprod_{i+j=n}c_{i,j}"] & \mathbf{X}_n
\end{tikzcd}$$

where $c_{i,j}$ is the adjoint of the assembly map

$$\gamma_{i,j}:T^{\otimes i}\otimes\mathbf{X}_j\to\mathbf{X}_n$$

defined in Def.4.3.6; Ex.4.3.1; 4.3.5, p.499.

$$\left(\begin{array}{c}\hom_{\Sigma_n}\left(\mathrm{Ind}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}(T^{\otimes i}\otimes\mathbf{X}_j),\mathbf{X}_n\right)\cong\hom_{\Sigma_i\times\Sigma_j}\left(T^{\otimes i}\otimes\mathbf{X}_j,\mathrm{Res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}\mathbf{X}_n\right)\\ c_{i,j}\mapsto\gamma_{i,j}\end{array}\right)$$

$\underline{\mathbf{Mod}_g(S^T)\to\mathbf{Spect}^{\Sigma}_T(\mathcal{C})}$ Given $(M_n)_{n\in\mathbb{N}}\in\mathbf{Mod}_g(S^T)$, take

$$\mathbf{M}=(M_n)_{n\in\mathbb{N}}\in\mathbf{Suite}(\Sigma,\mathcal{C})\leftarrow\mathbf{Spect}^{\Sigma}_T(\mathcal{C})$$

where the assembly map on degree n

$$\gamma_{n-1}:T\otimes M_{n-1}\to M_n$$

is the adjoint of the component $c_{1,n-1}$ of the degree n part of the S^T-action morphism:

$$\begin{array}{c}\hom_{\Sigma_n}\left(\mathrm{Ind}^{\Sigma_n}_{\Sigma_1\times\Sigma_{n-1}}(T\otimes M_{n-1}),M_n\right)\cong\hom_{\Sigma_1\times\Sigma_{n-1}}\left(T\otimes M_{n-1},\mathrm{Res}^{\Sigma_n}_{\Sigma_1\times\Sigma_{n-1}}M_n\right)\\ c_{1,n-1}\mapsto\gamma_{n-1}\end{array}$$

□

- Since S^T is commutative, we also have an isomorphism:

$$\mathbf{Spect}^{\Sigma}_T\simeq\mathbf{Mod}_d(S^T).$$

The unital symmetric monoidal product $-\otimes-$ on $\mathbf{Spect}_T^\Sigma(\mathcal{C})$ - 2 [Ayoub, p.503; Prop.4.3.71, p.503]

- Since S^T is a unital symmetric algebra,

The category $\mathbf{Mod}_g(S^T)$ is still unital symmetric monoidal:

If $\mathbf{X}$ and $\mathbf{Y}$ are two S^T-modules, we define $\mathbf{X}\otimes\mathbf{Y}$ as the usual coequalizer:

$$\mathbf{X}\otimes\mathbf{Y} := \mathsf{Coeq}\left(\mathbf{X}\otimes S^T\otimes\mathbf{Y} \rightrightarrows \mathbf{X}\otimes\mathbf{Y}\right),$$

where $\mathbf{X}$ is considered as a right S^T-module via the isomorphism $\mathbf{Mod}_g(S^T)\simeq\mathbf{Mod}_d(S^T)$.

- Suppose $\mathcal{C}$ is left (resp. right) closed and admits the relevant limits. then $\mathbf{Mod}_g(S^T)$ is also closed.

If $\mathbf{X}$ and $\mathbf{Z}$ are two S^T-modules, we define $\underline{\mathrm{Hom}}_g^{S^T}(\mathbf{X},\mathbf{Z})$ as the equalizer:

$$\underline{\mathrm{Hom}}_g^{S^T}(\mathbf{X},\mathbf{Z}) := \mathsf{Eq}\left(\underline{\mathrm{Hom}}_g(\mathbf{X},\mathbf{Z}) \rightrightarrows \underline{\mathrm{Hom}}_g(\mathbf{X}\otimes S^T,\mathbf{Z})\right)$$

where the first arrow is that deduced from the structural morphism $\mathbf{X}\otimes S^T\to\mathbf{X}$ of $\mathbf{X}$ viewed as right S^T-module and the second morphism is the composite:

$$\underline{\mathrm{Hom}}_g(\mathbf{X},\mathbf{Z}) \to \underline{\mathrm{Hom}}_g(\mathbf{X}\otimes S^T,\mathbf{Z}\otimes S^T) \to \underline{\mathrm{Hom}}_g(\mathbf{X}\otimes S^T,\mathbf{Z}).$$

We also note that the left S^T-module structure on $\underline{\mathrm{Hom}}_g^{S^T}(\mathbf{X},\mathbf{Z})$ is given by that of $\mathbf{Z}$.

- We thus obtain the unital symmetric monoidal category $(\mathbf{Spect}_T^\Sigma(\mathcal{C}),\otimes)$, which is also closed if $\mathcal{C}$ is closed and admits relevant limits.

- Set:

$$\begin{aligned}\underline{\mathrm{SUS}}_{T,\Sigma} : \mathbf{Suite}(\Sigma,\mathcal{C}) &\to \mathbf{Spect}_T^\Sigma(\mathcal{C})\\ X=(X_n)_{n\in\mathbb{N}} &\mapsto \coprod_{p\in\mathbb{N}} \underline{\mathrm{Sus}}_{T,\Sigma}^p(X_p),\end{aligned}$$

which functor corresponds via the isomrophism $\mathbf{Spect}_T^\Sigma(\mathcal{C})\simeq\mathbf{Mod}_g(S^T)$ the "associated free S^T-module functor" which to the symmetric sequence $X=(X_n)_{n\in\mathbb{N}}$ associate the left S^T-module $S^T\otimes X$. In particular, the functor $\underline{\mathrm{SUS}}_{T,\Sigma}$ is monoidal.

Proof. • The functor $\underline{\mathrm{SUS}}_{T,\Sigma}$ is left adjoint to the forgetful functor $\mathbf{Spect}_T^\Sigma(\mathcal{C})\to\mathbf{Suite}(\Sigma,\mathcal{C})$.

• The fucntor $S^T\otimes-$ is left adjoint to the forgetful functor $\mathbf{Mod}_g(S^T)\to\mathbf{Suite}(\Sigma,\mathcal{C})$.
The result is now clear. □

[Ayoub, Prop,4.3.75, p.504]

For a unital symmetric monoidal model category $\mathfrak{M}=(\mathfrak{M},\otimes,\mathbf{1})$, the category $\mathbf{Spect}_T^\Sigma(\mathfrak{M})$ is a monoidal model category for its unstable projective structure.

[Ayoub, Th,4.3.76, p.505]

Suppose that the category $\mathbf{Spect}_T^{\Sigma}(\mathfrak{M})$ is stable for its stable projective structure. Then

$$\left(\mathbf{Spect}_T^{\Sigma}(\mathfrak{M}), \mathbf{W}_{st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-st}\right)$$

is a monoidal model category.

[Ayoub, An application of Th,4.3.76 shall be found in p.540]

The category $\mathbb{SH}_{\mathfrak{M}}^{T}(\mathcal{F}, \mathcal{I})$ is a monoidal model category.

[Ayoub, Prop,4.3.77, Rem. 4.3.78, p.506]

The model category

$$\left(\mathbf{Spect}_T^{\Sigma}(\mathfrak{M}), \mathbf{W}_{st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-st}\right)$$

is stable, if and only if there exists an object $T' \in \mathrm{Ob}(\mathfrak{M})$ such that T is isomorphic to $\Sigma^1(T')$ in $\mathbf{Ho}(\mathfrak{M})$.

[Ayoub, An application of Prop,4.3.77 shall be found in p.540]

The category $\mathbb{SH}_{\mathfrak{M}}^{T}(\mathcal{F}, \mathcal{I})$ is triangulated.

[Ayoub, Th.4.3.79 p.506]

For the left Quillen functor

$$(- \otimes_{\{1\}} \Sigma) : \mathbf{Spect}_T(\mathfrak{M}) \to \mathbf{Spect}_T^{\Sigma}(\mathfrak{M})$$

to be a Quillen equivalence w.r.t. stable projective structures, it suffices that the following conditions are satisfied:

1. The permutation $(123) \in \Sigma_3$ operates by the identity on $T^{\otimes 3}$ in $\mathbf{Ho}(\mathfrak{M})$.
2. The functor $\mathrm{R}\underline{\mathrm{Hom}}_g(T, -)$ commutes with the transfinite composition as in the hypothesis 4.3.56.
3. The model category $\left(\mathbf{Spect}_T^{\Sigma}(\mathfrak{M}), \mathbf{W}_{st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-st}\right)$ is stable.

[Ayoub, An application of Th,4.3.79 shall be found in p.558]

The category $\mathbf{SH}_{\mathfrak{M}}(X)$ is equivalent to the stable homotopy category of non-symmetric T_X-spebtra.

3.5 Model theoretical approach to sheaves on general $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$

~~[Ayoub~~b, 4.4.3. Def.4.4.2 p.510; Rem.4.4.3, Prop.4.4.4, Prop.4.4.5, p.511; Def.4.4.7 p.512]

- $\mathcal{S}$, a small category, $F \in \mathbf{PreShv}(\mathcal{S}, Sets)$,
  $\mathcal{C}$, a cocomplete category, $K, H \in \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$

  $\Longrightarrow$ we get new objectts in $\mathbf{PreShv}(\mathcal{S}, \mathcal{C})$:

$$F \otimes K = \begin{pmatrix} \mathcal{S}^{op} \xrightarrow{F \otimes K} \mathcal{C} \\ U \mapsto F(U) \otimes K(U) \end{pmatrix} \in \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$$

$$\underline{\mathsf{hom}}(F, H) = \begin{pmatrix} \mathcal{S}^{op} \xrightarrow{\underline{\mathsf{hom}}(F,H)} \mathcal{C} \\ U \mapsto \underline{\mathsf{hom}}_{\mathcal{C}}(F \times U, H) \end{pmatrix} \in \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$$

  Here in general,

$$\underline{\mathsf{hom}}_{\mathcal{C}}(F, H) := \mathbf{Lim}_{(U \to F) \in \mathrm{Ob}(\mathcal{S}/F)}\, H(U) \in \mathrm{Ob}(\mathcal{C})$$

- (Note: For any $U \in \mathrm{Ob}(\mathcal{S})$, it appears Ayoub calls an object of $\mathbf{PreShv}(\mathcal{S})/U$ a sub-presheaf of sets of $U \in \mathrm{Ob}(\mathcal{S})$)
- For $U \in \mathrm{Ob}(\mathcal{S})$, we have a canonical isomorphism

$$\underline{\mathsf{hom}}_{\mathcal{C}}(U, H) \simeq H(U)$$

- Let F be a presheaf of sets. The functor

$$\underline{\mathsf{hom}}_{\mathcal{C}}(F, -) : \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \to \mathcal{C}$$

  is right adjoint to the composite functor:

$$\mathcal{C} \xrightarrow{cst} \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \xrightarrow{F \otimes -} \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$$

  where cst is the functor which associates to $A \in \mathrm{Ob}(\mathcal{C})$ the constant presheaf A_{cst} (i.e. s.t. $A_{cst}(U) = A$ for $U \in \mathrm{Ob}(\mathcal{S})$).
- Let F be a presheaf of sets. The endofunctor $\underline{\mathsf{hom}}(F, -)$ of $\mathbf{PreShv}(\mathcal{S}, \mathcal{C})$ is right adjoint of the endofunctor $F \otimes -$.
- Let F and G be two presheaves of sets on $\mathcal{S}$. There exists a natural isomorphism:

$$\underline{\mathsf{hom}}(F \times G, H) \simeq \underline{\mathsf{hom}}\,(G, \underline{\mathsf{hom}}(F, H))$$

  on $H \in \mathrm{Ob}\,(\mathbf{PreShv}(\mathcal{S}, \mathcal{C}))$.
- A Grothendieck topology top on a small category $\mathcal{S}$ associates to any $U \in \mathrm{Ob}(\mathcal{S})$ a family

$$J_{top}(U) \quad (\subseteq \mathrm{Ob}\,(\mathbf{PreShv}(\mathcal{S})/U))$$

  of sub-presheaves of sets of U such that the following conditions are datisfied:
  - $U \in J_{top}(U)$.
  - For any arrow $V \to U$ and any $R \in J_{top}(U)$, the sub-presheaf $R \times_U V \subset V$ is in $J_{top}(U)$.
  - Let $R \in J_{top}(U)$ and P a sub-presheaf of U. If for any $V \to R \in \mathrm{Ob}(\mathcal{S}/R)$ the sub-presheaf $V \times_U P$ is in $J_{top}(V)$, then $P \in J_{top}(U)$.

  The sub-presheaves in $J_{top}(U)$ are called (sieve blanket) of U for the topology top. The pair $(\mathcal{S}, top)$ is called a Grothendieck site .
- Suppose $\mathcal{S}$ is equipped with a topology top. $H \in \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$, a presheaf valued in $\mathcal{C}$ is a sheaf (resp. seperated) with respect to the topology top when the arrow

$$H(U) = \underline{\mathsf{hom}}_{\mathcal{C}}(U, H) \to \underline{\mathsf{hom}}_{\mathcal{C}}(R, H)$$

  is an isomorphism (resp. monomorphism) for any $U \in \mathrm{Ob}(\mathcal{S})$ and any sieve blanket R of U.
- Denote by $\mathbf{Shv}_{top}(\mathcal{S}, \mathcal{C})$ the full subcategory of $\mathbf{PreShv}(\mathcal{S}, \mathcal{C})$ whose objects are sheaves.

[Ayoub, p.512; Prop.4.4.8, p.512; Lem.4.4.9 Th.4.4.10, p.513; Cor.4.4.11, p.514]

- For $H \in \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$, following [SGA], we set:

$$\begin{cases} \mathrm{L}H(U) & = & \mathrm{Colim}_{R \in J_{top}(U)} \underline{\hom}_{\mathcal{C}}(R, H) \\ l(H) : H & \to & \mathrm{L}H \end{cases}$$

- Suppose that the monomorphisms of $\mathcal{C}$ are stable by small filtered colimits. Let H be a preshaf valued in $\mathcal{C}$. Then, H is a sheaf if and only if $l(H)$ is invertible: $l(H) : H \xrightarrow{\cong} \mathrm{L}H$.
- For $H \in \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$, the arrows:

$$\mathrm{L}(l(H)), l(\mathrm{L}(H)) : \mathrm{L}(H) \rightrightarrows \mathrm{L}\,(\mathrm{L}(H))$$

are equal.
- Suppose that $\mathcal{C}$ is presentable in the sense of the definition 4.2.16.

(Th.4.4.10) The inclusion

$$\mathbf{Shv}_{top}(\mathcal{S}, \mathcal{C}) \subset \mathbf{PreShv}(\mathcal{S}, \mathcal{C})$$

admits a left adjoint:

$$a_{top} : \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \to \mathbf{Shv}_t(\mathcal{S}, \mathcal{C})$$

Furthermore, the functor a_{top} commutes with finite limits.

(Cor.4.4.11) $\mathbf{PreShv}(\mathcal{S}, \mathcal{C})$ and $\mathbf{Shv}_t(\mathcal{S}, \mathcal{C})$ are also presentable.

Outline of the proof of Th.4.4.10.

– For any ordinals $\nu \leq \lambda$, define by transfinie induction:

$$\mathrm{L}^\lambda : \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \circlearrowleft, \qquad l_{\nu \leq \lambda} : \mathrm{L}^\nu \dashv \mathrm{L}^\lambda$$

$\underline{\lambda = 0}$ $\mathrm{L}^0 = \mathrm{id}$ and $l_{\lambda \leq \lambda} = \mathrm{id}$.
$\underline{\lambda \textbf{ is limit}}$ $\mathrm{L}^\lambda = \mathrm{Colim}_{\nu \in \lambda} \mathrm{L}^\nu$. For $\mu \in \lambda$,
$l_{\mu \leq \lambda} : \mathrm{L}^\nu \to \mathrm{Colim}_{\nu \in \lambda} \mathrm{L}^\nu = \mathrm{L}^\lambda$ the resulting evident map.
$\underline{\lambda = \nu + 1}$ $\mathrm{L}^\lambda = \mathrm{L} \circ \mathrm{L}^\nu$. For $\mu \in \lambda$, we take $l_{\mu \leq \lambda} = l(\mathrm{L}^\nu) \circ l_{\mu \leq \nu}$.

– Suppose that $\mathcal{C}$ is α-presentable with α larger than the cardinal of $\mathcal{S}$. By Lemma 4.4.9 and Proposition 4.4.8, we see we may take

$$a_{t_\emptyset} := \mathrm{L}^\lambda : \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \to \mathbf{Shv}_t(\mathcal{S}, \mathcal{C})\ (\subset \mathbf{PreShv}(\mathcal{S}, \mathcal{C}))$$

□

Outline of the proof of Cor.4.4.11.
This immediately follows from Proposition 4.2.20 and Proposition 4.2.21.

□

[Ayoub, Def.4.4.15, Prop.4.4.16, p.515; Prop.4.4.17, p.516; Def.4.4.18, p.517]

Let $\mathfrak{M}$ be a model category.

- Let $f; H \to K$ be an aroow of $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$.
  - We say f is a weak equivalence (resp. injective cofibration , projective fibration) when for any $U \in \mathrm{Ob}(\mathcal{S})$, the arrow $f(U); H(U) \to K(U)$ is a weak equivalence (resp. cofibration, fibration) of $\mathfrak{M}$.
  - We say f is a projective cofibration (resp. injective fibration) when f admits the left (resp. right) lifting property with respect to the trivial projective fibrations (resp. trivial injective cofibrations).
  - We denote classes of arrows of $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ as follows:
    - $\mathbf{W}$, the class of weak equivalences.
    - $\mathbf{Cof}_{proj}$ (resp. $\mathbf{Cof}_{inf}$), the class of projective (resp. injective) cofibrations.
    - $\mathbf{Fib}_{proj}$ (resp. $\mathbf{Fib}_{inj}$), the class of projective (resp. injective) fibrations.
- Suppose $\mathfrak{M}$ is presentable by cofibrations.
  - $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ with the structures $(\mathbf{W}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj})$ becomes a model category, called projective model structure , which is presentable by cofibrations.
  - $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ with the structures $(\mathbf{W}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{inj})$ becomes a model category, called injective model structure , which is presentable by cofibrations.

[Ayoub, Def.4.4.22, Def.4.4.23, p.517; Ex.4.4.24, Def.4.4.26, Rem.4.4.27, p.518]

- Let $\mathfrak{M}$ be a stable model category. An object $A \in \mathfrak{M}$ is called homotopically compact if for any $n \in \mathbb{Z}$, the functor
$$\hom_{\mathbf{Ho}(\mathfrak{M})}(A, -[n]) : \mathfrak{M} \to Sets$$
commute with filered small colimits.
- A model category $\mathfrak{M}$ is called a category of coefficients when the following conditions are satisfied:
  - $\mathfrak{M}$ is left proper, presentable by cofibrations, and stable.
  - Weak equivalences of $\mathfrak{M}$ are stable by finite coproducts.
  - There exists a set $\mathcal{E} \subset \mathfrak{M}$ of homotopically compact objects which generates the triangulated category $\mathbf{Ho}(\mathfrak{M})$ (see Theorem 4.1.49) equipped with infinite coproduct.
- The two model categories
$$\mathbf{Spect}_{S^1}(\Delta^{op}Set), \quad \mathbf{Spect}^{\Sigma}_{S^1}(\Delta^{op}Set)$$
provided with their stable projective structures aer categories of coefficients. The third condition follows from the fact that a filtered colimit of fibrant objects is once again a fibrant object.
Similary, if A is a ring, the category of complexes of left A-modules $\mathbf{COmpl}(\mathcal{M}od_g(A))$ is athe category of coefficients.
- Let H and K be two objects of $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$. We denote by $\underline{\Pi}_0(H, K)$ the presheaf of sets defined by
$$\begin{aligned} \underline{\Pi}_0(H,K) : \mathcal{S}^{op} &\to Sets \\ U &\mapsto \hom_{\mathbf{Ho}(\mathbf{PreShv}(\mathcal{S},\mathfrak{M}))}\left(U \overset{\mathsf{L}}{\otimes} H, K\right) \end{aligned}$$
Here, the derived functor $U \overset{\mathsf{L}}{\otimes} -$ is taken w.r.t.the injective structure on $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ considering U as a representable set-valued presheaf.
- The associated sheaf of $\underline{\Pi}_0(H, K)$ for the topology top is denoted by $\underline{\Pi}_0^{top}(H, K)$.
- When $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M}) \ni H = A_{cst}$, the constant functor at $A \in \mathrm{Ob}(\mathfrak{M})$, the presheaf of sets $\underline{\Pi}_0(H, K)$ is given by:
$$\begin{aligned} \underline{\Pi}_0(A_{cst},K) : \mathcal{S}^{op} &\to Sets \\ U &\mapsto \hom_{\mathbf{Ho}(\mathbf{PreShv}(\mathcal{S},\mathfrak{M}))}\left(U \overset{\mathsf{L}}{\otimes} A_{cst}, K\right) \\ &\cong \hom_{\mathbf{Ho}(\mathfrak{M})}(A, K(U)) \end{aligned}$$
We shall simply denote
$$\Pi_0(A, K) := \underline{\Pi}_0(A_{cst}, K)$$
whose associated sheaf by $\Pi_0^{top}(A, K)$.

[Ayoub, Def.4.4.28, Prop.4.4.32, p.518, Def.4.4.34, Lem.4.4.35, p.520]

- Suppose $\mathfrak{M}$ is a category of coefficients. An arrow $f : H \to K$ of $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ is called a <u>*top*-local equivalence</u> if for any $n \in \mathbb{Z}$ and $A \in \mathcal{E}$, the sheaf morphism

$$\Pi_0^{top}(A, H[n]) \to \Pi_0^{top}(A, K[n])$$

is invertible.
- We denote by $\mathcal{L}_{top}$ (resp. $\mathcal{L}_{top,\beta}$) the class of top-local (resp. those with whose source and target β-accessible) equivalences.
- If $\mathfrak{M}$ is a category of coefficients, there exists a cardinal β such that

$$\mathbf{W}_{\mathcal{L}_{top}} = \mathbf{W}_{\mathcal{L}_{top,\beta}}.$$

Furthermore,

$$\mathcal{L}_{top} = \mathbf{W}_{\mathcal{L}_{top}}$$

- Suppose $\mathfrak{M}$ is a category of coefficients. The <u>*top*-local projective (resp. injective) structure</u> $(\mathbf{W}_{top}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-top})$ (resp. $(\mathbf{W}_{top}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-top})$) on $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ is the Bousfield localisation of the projective (respl injective) structure of Definition 4.4.18, with respect to $\mathcal{L}_{top}$. The homotopy category of these model structures are denoted by $\mathbf{Ho}_{top}(\mathbf{PreShv}(\mathcal{S}, \mathfrak{M}))$.
- By the second part of Proposition 4.4.32, $\mathbf{W}_{top}$ is exactly the class of *top*-local equivalences.
- (The following Lemma 4.4.35 indicates that for

$$\mathfrak{M} = \mathbf{Spect}_{S^1}(\Delta^{op}Set),\ \mathbf{Spect}^{\Sigma}_{S^1}(\Delta^{op}Set)$$

as in Example 4.4.24, we may regard

$$\mathbf{PreShv}(\mathcal{S}, \mathfrak{M}) \approx \mathbf{Spect}_{S^1}(\mathbf{PreShv}(\mathcal{S}, Set)),\ \mathbf{Spect}^{\Sigma}_{S^1}(\mathbf{PreShv}(\mathcal{S}, Set))$$

with

$$\mathbf{Ho}(\mathbf{PreShv}(\mathcal{S}, \mathfrak{M}))$$

a triangulated category by Theorem 4.1.49):

Ayoub, Lem.4.4.35, p.520

Under the hypothesis of Definition 4.4.34 above, the *top*-local model structure on

$$\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$$

is left proper and stable.

[Ayoub, Def. 4.4.41, Lem.4.4.42, p.523; Cor.4.4.43, p.524]

- Let f be an arrow of $\mathbf{Shv}_{top}(\mathcal{S}, \mathfrak{M})$.
  - f is called a <u>weak equivalence</u> once regarded *top*-local weak equivalence in $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$. Denote the class of weak equivalences in $\mathbf{Shv}_{top}(\mathcal{S}, \mathfrak{M})$ by $\mathbf{W}$.
  - f is called a <u>projective (resp. injective) fibration</u> once regarded *top*-local projective (resp. injective) fibration in $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$.
    Denote the class of projective (resp. injective) fibrations in $\mathbf{Shv}_{top}(\mathcal{S}, \mathfrak{M})$ by $\mathbf{Fib}_{proj}$ (resp. $\mathbf{Fib}_{inj}$).
  - f is called a <u>projective (resp. injective) cofiration</u> when it admits a left lifting property with respect to trivial projective (resp. injective) fibrations.
    Denote the class of projective (resp. injective) cofibrations in $\mathbf{Shv}_{top}(\mathcal{S}, \mathfrak{M})$ by $\mathbf{Cof}_{proj}$ (resp. $\mathbf{Cof}_{inj}$).
- Both $(\mathbf{W}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj})$ and $(\mathbf{W}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{inj})$ define model category structures which are presentable by cofibrations on $\mathbf{Shv}_{top}(\mathcal{S}, \mathfrak{M})$.
- The latter is Quillen equivalent to $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ equipped with the *top*-local structures. This follows from the following general result:

[Ayoub, Lemma 4.4.42, p.523]

Let $(F, G) : \mathcal{C} \to \mathcal{D}$ be an adjoint pair with $\mathcal{C} = (\mathcal{C}, \mathbf{W}, \mathbf{Cof}, \mathbf{Fib})$ a model category which is α-presentably by cofibrations and $\mathcal{D}$ a bicomplete category. Suppose the following conditions are satisfied:

- G is fully faithful and α-accessible.
- F commutes with finie limits.
- The functor $G \circ F$ preserve β-accessible objects for $\beta \geqq \alpha$.
- For any $A \in \mathrm{Ob}(\mathcal{C})$, the unit
  $A \to G \circ F(A)$ is a weak equivalence.

Then $(\mathcal{D}, G^{-1}\mathbf{W}, F\,\mathbf{Cof}, G^{-1}\mathbf{Fib})$ is a model category, which is α-presentable by cofibrations. Furhtermore, (F, G) becomes a Quillen equivalence.

[Ayoub, 4.4.3. Lem.4.4.44 p.524]

- Given $\mathcal{C}$, a bicomplete category, $f : \mathcal{S} \to \mathcal{S}'$, a functor between small categories,

$$\Longrightarrow \quad \begin{aligned} f_* : \mathbf{PreShv}(\mathcal{S}', \mathcal{C}) &\to \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) \\ H' &\mapsto H' \circ f \end{aligned}$$

- f_* admits a left adjoint

$$\begin{aligned} f^* : \mathbf{PreShv}(\mathcal{S}, \mathcal{C}) &\to \mathbf{PreShv}(\mathcal{S}', \mathcal{C}) \\ K &\mapsto \left(f^*K : \mathrm{Ob}(\mathcal{S}') \ni U' \mapsto \operatorname*{Colim}_{(U' \to f(U)) \in \mathrm{Ob}(U' \backslash \mathcal{S})} K(U) \in \mathrm{Ob}(\mathcal{C}) \right) \end{aligned}$$

[Ayoub, Def. 4.4.47, Def.4.4.50, p.525]

Suppose we are given two Grothendieck sites $(\mathcal{S}, top)$ and $(\mathcal{S}', top')$.

- A functor $f : \mathcal{S} \to \mathcal{S}'$ is called continuous , when for all sheaf of sets F' on $(\mathcal{S}', top')$, the presheaf $F' \circ f : \mathcal{S}^{op} \to Set$ is a sheaf on $(\mathcal{S}, top)$.
- A pre-morphism of sites $f : (\mathcal{S}', top') \to (\mathcal{S}, top)$ is a continuous functor $f : \mathcal{S} \to \mathcal{S}'$.
- Given a pre-morphism of sites $f : (\mathcal{S}', top') \to (\mathcal{S}, top)$,

$$f_* : \mathbf{PreShv}(\mathcal{S}') \to \mathbf{PreShv}(\mathcal{S})$$

  restricts to

$$f_* : \mathbf{Shv}_{top'}(\mathcal{S}') \to \mathbf{Shv}_{top}(\mathcal{S}),$$

  called the direct image functor of sheaves, which admits a left adjoint

$$f^* := a_{top'} \circ f^*_{presheaves} \circ incl :$$
$$\mathbf{Shv}_{top}(\mathcal{S}) \xrightarrow{incl} \mathbf{PreShv}_{top}(\mathcal{S}) \xrightarrow{f^*_{presheaves}}$$
$$\mathbf{PreShv}_{top'}(\mathcal{S}') \xrightarrow{a_{top'}} \mathbf{Shv}_{top'}(\mathcal{S}')$$

- A pre-morphism of sites $f : (\mathcal{S}', top') \to (\mathcal{S}, top)$ is called a morphism of sites when the functor $f^* : \mathbf{Shv}_{top}(\mathcal{S}) \to \mathbf{Shv}_{top'}(\mathcal{S}')$ commutes with finite limits.
- A pre-morphism of sites $f : (\mathcal{S}', top') \to (\mathcal{S}, top)$ is called a pseudo-morphism of sites when the functor $f^* : \mathbf{Shv}_{top}(\mathcal{S}) \to \mathbf{Shv}_{top'}(\mathcal{S}')$ commutes with fiber products and equalizers.

[Ayoub, Th.4.4.51 p.526]

Let $\mathfrak{M}$ be a coefficient category . Let $f : (\mathcal{S}', top') \to (\mathcal{S}, top)$ be a pseudo-morphism of sites . Then the adjunction

$$(f^*, f_*) : \mathbf{PreShv}(\mathcal{S}, \mathfrak{M}) \to \mathbf{PreShv}(\mathcal{S}', \mathfrak{M})$$

be a Quillen adjunction w.r.t. projective local structure of the definition 4.4.34. Furthermore, the functor f^* sends local-top equivalences to local-top' equivalences .

[Ayoub, Def.4.4.60, p.530; Th.4.4.61 p.531]

- Let $(\mathcal{S}, top), (\mathcal{S}', top')$ be two Grothendieck sites provided with P-structures $\mathcal{E}, \mathcal{E}'$. Then a premorphism of sites $f : (\mathcal{S}', top') \to (\mathcal{S}, top)$ is called compatible with P-structures , when for any $U \in \mathrm{Ob}(\mathcal{S})$:
  - the functor $f_U : \mathcal{S}/U \to \mathcal{S}'/f(U)$ sends $\mathcal{E}_U$ in $\mathcal{E}'_{f(U)}$ and induces a functor $f_U^{\mathcal{E}} : \mathcal{E}_U \to \mathcal{E}'_{f(U)}$,
  - the functor $f_U^{\mathcal{E}}$ defines a peudomorphism of sites $f_U^{\mathcal{E}} : \left(\mathcal{E}'_{f(U)}, top'_{f(U)}\right) \to (\mathcal{E}_U, top)$.
- Given a premorphism of sites $f : (\mathcal{S}', top') \to (\mathcal{S}, top)$, which is compatible with two P-structures on $\mathcal{S}$ and $\mathcal{S}'$ as in the definition 4.4.60. Then

$$(f^*, f_*) : \mathbf{PreShv}(\mathcal{S}, \mathfrak{M}) \to \mathbf{PreShv}(\mathcal{S}', \mathfrak{M})$$

is a Quillen adjunction w.r.t. the projective local structures.

[Ayoub, Prop.4.4.62 p.531]

Let $\mathfrak{M}$ be a (resp. symmetric) monoidal model category. Suppose that $\mathfrak{M}$ is presentable. Then the category $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ is naturally a (resp. symmetric) monoidal model category w.r.t. prejective and injective structures of the definition 4.4.18 .

[Ayoub, Prop.4.4.63 p.531]

Suppose that the coefficient category $\mathfrak{M}$ is also a (resp. symmetric) monoidal model category. If the site $(\mathcal{S}, top)$ admits enough points. Then $\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})$ is a (resp. symmetric) monoidal model category w.r.t. prejective and injective top-local structures .

3.6 Model categories on PreShv $((\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M})$ and $(f,\alpha)_\sharp \dashv (f,\alpha)^* \dashv (f,\alpha)_*$

[Ayoub, 4.5.1, Def.4.5.1, Lem.4.5.2, Prop.4.5.3, p.532]

- Given a diagram $(\mathcal{F},\mathcal{I})$ of S-schemes, We denote $\mathrm{Sm}/(\mathcal{F},\mathcal{I})$ the category:
  - an object is of the form
$$((U \to \mathcal{F}(i)) \in \mathrm{Sm}/\mathcal{F}(i), i \in \mathrm{Ob}(\mathcal{I})),$$
which shall be simply denoted by (U,i).
  - an arrow $(U';i') \to (U,i)$ is a couple $(U' \to U, i' \to i)$ such that the following square commutes:
$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{F}'(i') \arrow[r] & \mathcal{F}(i) \end{tikzcd}$$

- A 1-morphism $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ of **Dia Sch** $/S$ yields a canonical factorisation:
$$(\mathcal{G},\mathcal{J}) \xrightarrow{f} (\mathcal{F}\circ\alpha,\mathcal{J}) \xrightarrow{\alpha} (\mathcal{F},\mathcal{I})$$

- f induces a functor
$$\begin{aligned} f = - \times_{\mathcal{F}} \mathcal{G} : \mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}) &\to \mathrm{Sm}/(\mathcal{G},\mathcal{J}) \\ ((V \to \mathcal{F}(\alpha(j))), j) &\mapsto ((V \times_{\mathcal{F}(\alpha(j))} \mathcal{G}(j), j)), \end{aligned}$$
which in turn induces an adjunction (c.f. Lem.4.4.44)
$$(f^*, f_*) : \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}),\mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{G},\mathcal{J}),\mathfrak{M})$$

- α induces a functor
$$\begin{aligned} \overline{\alpha} : \mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}) &\to \mathrm{Sm}/(\mathcal{F},\mathcal{I}) \\ ((U \to \mathcal{F}(\alpha(j))), j) &\mapsto ((U \to \mathcal{F}(\alpha(j))), \alpha(j)) \end{aligned}$$
which in turn induces the functor (c.f. Lem.4.4.44)
$$\alpha^* := \overline{\alpha}_* : \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}),\mathfrak{M})$$

- Define the functor
$(f,\alpha)^* := f^* \circ \alpha^* : \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{G},\mathcal{J}),\mathfrak{M})$
- Explicitly, for $H \in \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M})$, $(f,\alpha)^*H$ is given by the association:
$$\begin{aligned} (f,\alpha)^*H : \mathrm{Ob}(\mathrm{Sm}/\mathcal{G}(j)) &\to \mathrm{Ob}\,\mathfrak{M} \\ (V,j) &\mapsto \mathrm{Colim}_{(V \to U \times_{\mathcal{F}(\alpha(j))} \mathcal{G}(j)) \in \mathrm{Ob}(V \backslash (\mathrm{Sm}/\mathcal{F}(\alpha(j))))} H(U,\alpha(j)) \end{aligned}$$

- The association $(f,\alpha) \mapsto (f,\alpha)^*$ extends naturally to a contravariant 2-functor:
$$\begin{aligned} \mathbf{PreShv}(\mathrm{Sm}/(-,-),\mathfrak{M}) : \mathbf{Dia\,Sch}/S &\to \mathfrak{Cat} \\ (\mathcal{F},\mathcal{I}) &\mapsto \mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F}-,\mathcal{I}-),\mathfrak{M}) \\ \left((\mathcal{G},\mathcal{J}) \xrightarrow{(f,\alpha)} (\mathcal{F},\mathcal{I})\right) &\mapsto (f,\alpha)^* \end{aligned}$$

[Ayoub, Prop. 4.5.4, p.533]

Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schemes.

- The functor $(f,\alpha)^*$ admits a right adjoint $(f,\alpha)_*$.
- If (f,α) is smooth argument by argument, the functor $(f,\alpha)^*$ admits a left adjoint $(f,\alpha)_\sharp$.

[Ayoub, Def. 4.5.8, Prop.4.5.9, p.534]

- A morphism $f : H \to K$ of presheaves on $\mathrm{Sm}/(\mathcal{F},\mathcal{I})$ valued in $\mathfrak{M}$ is a <u>semi-projective cofibration</u> if for any $i \in \mathrm{Ob}(\mathcal{I})$, the functor

$$\begin{aligned}(\mathrm{id}_{\mathcal{F}(i)}, i)^* : &\mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}) \\ &\to \mathbf{PreShv}(\mathrm{Sm}/\mathcal{F}(i),\mathfrak{M}),\end{aligned}$$

  induced by

$$(\mathrm{id}_{\mathcal{F}(i)}, i) : \mathcal{F}(i) = (\mathcal{F}(i), *) \to (\mathcal{F},\mathcal{I}),$$

  sends the morphism f to a projective cofibration in $\mathbf{PreShv}(\mathrm{Sm}/\mathcal{F}(i),\mathfrak{M})$.
- The class of semi-projective cofibrations is denoted by $\mathbf{Cof}_{s-pr}$.
- Define the class $\mathbf{Fib}_{s-pr}$ of <u>semi-projective fibrations</u> by the right lifting property with respect to $\mathbf{Cof}_{s-pr} \cap \mathbf{W}$.
- The triplet $(\mathbf{W}, \mathbf{Cof}_{s-pr}, \mathbf{Fib}_{s-pr})$ defines the <u>semi-projective model structure</u> which is presentable by cofibrations on the category $\mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M})$.

[Ayoub, p.535; Th. 4.5.10, p.536]

- Set
$$\tau \in \{\mathrm{Nis}, \text{ét}\}$$
and equip $\mathrm{Sm}/(\mathcal{F},\mathcal{I})$ with the topology τ generated by the families of the form:
$$((u_\alpha, \mathrm{id}_i) : (U_\alpha, i) \to (U, i))_\alpha$$
where $(u_\alpha)_\alpha$ form a covering family of U for the topology τ.

- By Proposition 4.4.32, we can Bousfield localize the model structure $(\mathbf{W}, \mathbf{Cof}_{s-pr}, \mathbf{Fib}_{s-pr})$ of Prop.4.5.9 by τ-local equivalences to obtain the τ-local semi-projective model structure on $\mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{J}), \mathfrak{M})$ which we denote by $(\mathbf{W}_\tau, \mathbf{Cof}_{s-pr}, \mathbf{Fib}_{s-pr-\tau})$.

- Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schemes.
  - With respect to the τ-local semi-projective structures $(\mathbf{W}_\tau, \mathbf{Cof}_{s-pr}, \mathbf{Fib}_{s-pr-\tau})$. we have the following two Quillen adjunctions:
$$((f,\alpha)^*, (f,\alpha)_*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M}).$$
$$(f_\sharp, f^*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M})$$
where the latter is available when f is cartesian and smooth argument by argument.
  - With respect to the τ-local projective structures $(\mathbf{W}_\tau, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\tau})$. we have the following three Quillen adjunctions :
$$(f^*, f_*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M})$$
$$(\alpha_\sharp, \alpha^*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M})$$
$$((f,\alpha)_\sharp, (f,\alpha)^*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M})$$
where the last is available when (f,α) is smooth argument by argument.
  - With respect to the τ-local injective structures $(\mathbf{W}_\tau, \mathbf{Cof}_{inj}, \mathbf{Fib}_{injj-\tau})$. we have the following Quillen adjunction:
$$(\alpha^*, \alpha_*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M})$$

[Ayoub, p.538, Prop. 4.5.11, p.536; p.537, Def.4.5.12, Lem.4.5.13, p.537]

- Let $(\mathcal{F},\mathcal{I})$ be a diagram of S-schemes. Consider the class $\mathcal{A}$ of arrows of $\mathbf{PreShv}\left(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}\right)$ of the form
$$\sigma(U,i,B) : (U,i)\otimes B_{cst} \xrightarrow{s_0\otimes \mathrm{id}_B} (\mathbb{A}^1_U,i)\otimes B_{cst}$$
with $i\in \mathrm{Ob}(\mathcal{I}), U$, a smooth $\mathcal{F}(i)$-scheme, s_0 the zero section of the affine line and B a cofibrant object of $\mathfrak{M}$.

- Denote by $\mathcal{A}_\alpha$ the class of $\sigma(U,i,B)$ with B an α-accessible object.

- Suppose β is sufficiently large. Let $K\in \mathrm{Ob}\left(\mathbf{PreShv}\left(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}\right)\right)$ be fibrant for the structure $(\mathbf{W}_\tau,\mathbf{Cof}_{s-pr},\mathbf{Fib}_{s-pr})$. Then, K is $\mathcal{A}_\beta$-local if and only if for any $i\in\mathrm{Ob}(\mathcal{I})$ and a smooth $\mathcal{F}(i)$-scheme U, the arrow $K\left(\mathbb{A}^1_U,i\right)\to K(U,i)$ is a weak equivalence.

- Consequently, $(\mathbf{W}_\tau)_{\mathcal{A}_\beta} = (\mathbf{W}_\tau)_{\mathcal{A}}$, which shows that the Bousfield localisation with respect to $\mathcal{A}$ exists.

- Let $(\mathcal{F},\mathcal{I})$ be a diagram of S-schemes. The $\mathbb{A}^1$-local semi-projective (resp. projective, injective) model structure :
$$\left(\mathbf{W}_{\mathbb{A}^1},\mathbf{Cof}_{s-pr},\mathbf{Fib}_{s-pr-\mathbb{A}^1}\right)$$
$$\left(resp.\ \left(\mathbf{W}_{\mathbb{A}^1},\mathbf{Cof}_{proj},\mathbf{Fib}_{proj-\mathbb{A}^1}\right),\left(\mathbf{W}_{\mathbb{A}^1},\mathbf{Cof}_{inj},\mathbf{Fib}_{inj-\mathbb{A}^1}\right)\right)$$
is the Bousfield localisation with respect to $\mathcal{A}$ of the τ-local semi-projective (resp. projective, injective) model structure. The arrows in $\mathbf{W}_{\mathbb{A}^1}$ are called the $\mathbb{A}^1$-weak equivalences .

- Let $(\mathcal{F},\mathcal{I})$ be a diagram of S-schemes. We define $\mathbb{A}^1_{\mathcal{F},\mathcal{I}}\in\mathbf{PreShv}\left(\mathrm{Sm}/(\mathcal{F},\mathcal{I})\right)$ by:
$$\mathbb{A}^1_{\mathcal{F},\mathcal{I}} : \mathrm{Sm}/(\mathcal{F},\mathcal{I}) \to Sets$$
$$(U,i)\mapsto \hom_{\mathbf{Sch}/\mathcal{F}(i)}\left(U,\mathbb{A}^1_{\mathcal{F}(i)}\right)$$
  - Let $K\in\mathrm{Ob}\left(\mathbf{PreShv}\left(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}\right)\right)$ be a fibrant object w.r.t. the τ-local projective structure $(\mathbf{W}_\tau,\mathbf{Cof}_{proj},\mathbf{Fib}_{proj-\tau})$. Then, K is $\mathbb{A}^1$-local if and only if the arrow $\underline{\hom}\left(\mathbb{A}^1_{\mathcal{F},\mathcal{I}},K\right)\to K$ is a weak equivalence of presheaves.
  - Let $H\in\mathrm{Ob}\left(\mathbf{PreShv}\left(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}\right)\right)$ be an injective cofibrant presheaf. Then, the arrow $H\to\mathbb{A}^1_{\mathcal{F},\mathcal{I}}\otimes H$ induced by the null section is a weak $\mathbb{A}^1$-equivalence.

[Ayoub, Th. 4.5.14, p.537; Prop.4.5.16, p.538]

Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schemes.

- With respect to the $\mathbb{A}^1$-local semi-projective structures $(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{s-pr}, \mathbf{Fib}_{s-pr-\mathbb{A}^1})$. we have the following two Quillen adjunctions:

$$((f,\alpha)^*, (f,\alpha)_*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M}).$$
$$(f_\sharp, f^*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M})$$

  where the latter is available when f is cartesian and smooth argument by argument.

- With respect to the $\mathbb{A}^1$-local projective structures $(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\mathbb{A}^1})$. we have the following three Quillen adjunctions :

$$(f^*, f_*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M})$$
$$(\alpha_\sharp, \alpha^*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M})$$
$$((f,\alpha)_\sharp, (f,\alpha)^*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{G},\mathcal{J}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M})$$

  where the last is available when (f,α) is smooth argument by argument.

- With respect to the $\mathbb{A}^1$-local injective structures $(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{injj-\mathbb{A}^1})$. we have the following Quillen adjunction:

$$(\alpha^*, \alpha_*) : \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M}) \to \mathbf{PreShv}((\mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}), \mathfrak{M})$$

- For $K, L \in \mathrm{Ob}(\mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}), \mathfrak{M}))$, the natural arrow

$$(f,\alpha)^*(K \otimes L) \to (f,\alpha)^*(K) \otimes (f,\alpha)^*(L)$$

  is invertible. Thus, $(f,\alpha)^*$ is a monoidal functor.

3.7 $\mathbb{M}_T(\mathcal{F},\mathcal{I}) := \mathbf{Spect}^{\Sigma}_{T_{\mathcal{F},\mathcal{I}}}(\mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}))$, $(f,\alpha)_\sharp \dashv (f,\alpha)^* \dashv (f,\alpha)_*$

[Ayoub, Def. 4.5.18, p.539]

- Fix a projective cofibrant object T of $\mathbf{PreShv}(\mathrm{Sm}/S,\mathfrak{M})$ such that for any smooth S-scheme U, the functor
$$\underline{\mathrm{Hom}}(T(U),-):\mathfrak{M}\to\mathfrak{M}$$
is accessible.
- Given a diagram of S-schemes $(\mathcal{F},\mathcal{I})$, we set
$$T_{\mathcal{F},\mathcal{I}} := \pi^*_{\mathcal{F},\mathcal{I}}T \in \mathrm{Ob}\,(\mathbf{PreShv}\,(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}))$$
where
$$\pi_{\mathcal{F},\mathcal{I}}:(\mathcal{F},\mathcal{I})\to S$$
is the projection structure.
- The functor
$$\underline{\mathrm{Hom}}\,(T_{\mathcal{F},\mathcal{I}},-)$$
is accessible.
- We denote the category $\mathbb{M}_T(\mathcal{F},\mathcal{I})$ as follows:
$$\mathbb{M}_T(\mathcal{F},\mathcal{I}):=\mathbf{Spect}^{\Sigma}_{T_{\mathcal{F},\mathcal{I}}}(\mathbf{PreShv}\,(\mathbf{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}))\,.$$

[Ayoub, Prop. 4.5.19, p.539]

The 2-functor
$$\mathbf{PreShv}\,(\mathrm{Sm}/(-,-),\mathfrak{M}):\mathbf{Dia\,Sch}/S\to\mathfrak{Mono}$$
which to a 1-morphism (f,α) asssociate the monoidal functor $(f,\alpha)^*$ naturally induces a contravariant 2-functor:
$$\mathbb{M}_T(-,-):\mathbf{Dia\,Sch}/S\to\mathfrak{Mono}$$

[Ayoub, Def.4.5.21, p.540; p.543]

For a diagram of S-schems $(\mathcal{F},\mathcal{I})$, the semi-projective (resp. projective, injective) $\mathbb{A}^1$-local structure $(\mathbf{W}_{\mathbb{A}^1},\mathbf{Cof}_{s-pr},\mathbf{Fib}_{s-pr-\tau})$ (resp. $(\mathbf{W}_{\mathbb{A}^1},\mathbf{Cof}_{proj},\mathbf{Fib}_{proj-\tau})$, $(\mathbf{W}_{\mathbb{A}^1},\mathbf{Cof}_{inj},\mathbf{Fib}_{inj-\tau})$) on $\mathbf{PreShv}(\mathbf{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M})$ induces <u>$\mathbb{A}^1$-stable semi-projective (resp. projective, injective)</u> $(\mathbf{W}_{\mathbb{A}^1-st},\mathbf{Cof}_{s-pr},\mathbf{Fib}_{s-pr-\mathbb{A}^1-st})$ (resp. $(\mathbf{W}_{\mathbb{A}^1-st},\mathbf{Cof}_{proj},\mathbf{Fib}_{proj-\mathbb{A}^1-st})$, $(\mathbf{W}_{\mathbb{A}^1-st},\mathbf{Cof}_{inj},\mathbf{Fib}_{inj-\mathbb{A}^1-st})$) on

$$\mathbb{M}_T(\mathcal{F},\mathcal{I}) \overset{\text{Def.4.5.18}}{:=} \mathbf{Spect}^{\Sigma}_{T_{\mathcal{F},\mathcal{I}}}(\mathbf{PreShv}(\mathbf{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M})).$$

Arrows in $\mathbf{W}_{\mathbb{A}^1-st}$ are called <u>stable $\mathbb{A}^1$-equivalences</u> and

$$\mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F},\mathcal{I}) := \mathbf{Ho}(\mathbb{M}_T(\mathcal{F},\mathcal{I})).$$

For $X \in \mathbf{Sch}/S$, set

$$\mathbf{SH}^T_{\mathfrak{M}}(X) := \mathbb{SH}^T_{\mathfrak{M}}(X,\mathbf{e})$$

[Ayoub, Th. 4.5.23, p.540]

Let $(f,\alpha):(\mathcal{G},\mathcal{J})\to(\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schemes.

- With respect to the $\mathbb{A}^1$-stable semi-projective structures $(\mathbf{W}_{\mathbb{A}^1-st},\mathbf{Cof}_{s-pr},\mathbf{Fib}_{s-pr-\mathbb{A}^1-st})$. we have the following two Quillen adjunctions:

$$((f,\alpha)^*,(f,\alpha)_*):\mathbb{M}_T(\mathcal{F},\mathcal{I})\to\mathbb{M}_T(\mathcal{G},\mathcal{J}),$$
$$(f_\sharp,f^*):\mathbb{M}_T(\mathcal{G},\mathcal{J})\to\mathbb{M}_T(\mathcal{F}\circ\alpha,\mathcal{J})$$

  where the latter is available when f is cartesian and smooth argument by argument.

- With respect to the $\mathbb{A}^1$- stable projective structures $(\mathbf{W}_{\mathbb{A}^1-st},\mathbf{Cof}_{proj},\mathbf{Fib}_{proj-\mathbb{A}^1-st})$. we have the following three Quillen adjunctions :

$$(f^*,f_*):\mathbb{M}_T(\mathcal{F}\circ\alpha,\mathcal{J})\to\mathbb{M}_T(\mathcal{G},\mathcal{J}),$$
$$(\alpha_\sharp,\alpha^*):\mathbb{M}(\mathcal{F}\circ\alpha,\mathcal{J})\to\mathbb{M}_T(\mathcal{F},\mathcal{I}),$$
$$((f,\alpha)_\sharp,(f,\alpha)^*):\mathbb{M}_T(\mathcal{G},\mathcal{J})\to\mathbb{M}_T(\mathcal{F},\mathcal{I})$$

  where the last is available when (f,α) is smooth argument by argument.

- With respect to the $\mathbb{A}^1$-local injective structures $(\mathbf{W}_{\mathbb{A}^1},\mathbf{Cof}_{inj},\mathbf{Fib}_{injj-\mathbb{A}^1})$. we have the following Quillen adjunction:

$$(\alpha^*,\alpha_*):\mathbb{M}_T(\mathcal{F},\mathcal{I})\to\mathbb{M}_T(\mathcal{F}\circ\alpha,\mathcal{J})$$

Proof. These claims immediately follow from Theorem 4.5.14 and Lemma 4.3.34. □

Corollary

By taking the homotopy categories, we obtain the following adjunctions:

$$\mathbb{SH}^T_{\mathfrak{M}}(\mathcal{G},\mathcal{J}) \underset{\mathsf{R}(f,\alpha)^*}{\overset{\mathsf{L}(f,\alpha)_\sharp}{\rightleftarrows}} \mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F},\mathcal{I}) \underset{\mathsf{R}(f,\alpha)_*}{\overset{\mathsf{L}(f,\alpha)^*}{\rightleftarrows}} \mathbb{SH}^T_{\mathfrak{M}}(\mathcal{G},\mathcal{J})$$

3.8 Left Quillen equivalences $\mathbb{M}_T(\mathcal{F},\mathcal{I}) :=$ $\mathbf{Spect}^{\Sigma}_{T_{\mathcal{F},\mathcal{I}}}(\mathbf{PreShv}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M})) \xrightarrow{a_{t_\emptyset}} \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_{\mathcal{F},\mathcal{I}})}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}))$

[Ayoub, p.543, p.551]

- (In the framwork of Grothendieck topology recalled in p.512, Definition 4.4.7, p.512) We introduce the topology $t_\emptyset$ on Sm/X for which

$$\mathsf{J}_{t_\emptyset}(U/X) = \begin{cases} \{\mathrm{id}_{U/X} : (U/X) \to (U/X)\} & (U \neq \emptyset) \\ \{\mathrm{id}_{\emptyset/X} : (\emptyset/X) \to (\emptyset/X),\ \emptyset \to (\emptyset/X)\} & (U = \emptyset) \end{cases}$$

  where $\emptyset$ denotes the pressheaf of emptysets and $(\emptyset/X)$ denotes the empty X-scheme.
- $K \in \mathbf{PreShv}(\mathrm{Sm}/X,\mathcal{C})$ is in $\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathcal{C})$
  $\iff$ $K(\emptyset/X)$ is a final object of $\mathcal{C}$.
- For a presheaf F the associated sheaf functor $a_{t_\emptyset}$ (which is left adjoint to the inclusion

$$\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathcal{C}) \subset \mathbf{PreShv}(\mathrm{Sm}/X,\mathcal{C})\ )$$

  is given as follows:

$$\left(a_{t_\emptyset}F\right)(U) = \begin{cases} *_{\mathcal{C}} & (Y \simeq \emptyset/X) \\ F(U) & (Y \not\simeq \emptyset/X) \end{cases}$$

  where $*_{\mathcal{C}}$ denotes the final object of $\mathcal{C}$.
- When $\mathcal{C}$ is pointed, the inclusion

$$\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathcal{C}) \subset \mathbf{PreShv}(\mathrm{Sm}/X,\mathcal{C})\ )$$

  admits a right adjoint

$$\mathbf{b} : \mathbf{PreShv}(\mathrm{Sm}/X,\mathcal{C}) \to \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathcal{C})$$

  which associates to a presheaf F the $t_\emptyset$-sheaf $b(F)$ given by

$$(\mathbf{b}F)(U) = F(U) \times_{F(\emptyset/X)} *_{\mathcal{C}} \quad (U \in \mathrm{Ob}(\mathrm{Sm}/X))$$

- Recall the important:

  [Ayoub, Lemma 4.4.42, p.523]

  Let $(F,G) : \mathcal{C} \to \mathcal{D}$ be an adjoint pair with $\mathcal{C} = (\mathcal{C},\mathbf{W},\mathbf{Cof},\mathbf{Fib})$ a model category which is α-presentably by cofibrations and $\mathcal{D}$ a bicomplete category. Suppose the following conditions are satisfied:

  - G is fully faithful and α-accessible.
  - F commutes with finie limits.
  - The functor $G \circ F$ preserve β-accessible objects for $\beta \geqq \alpha$.
  - For any $A \in \mathrm{Ob}(\mathcal{C})$, the unit
    $A \to G \circ F(A)$ is a weak equivalence.

  Then $(\mathcal{D}, G^{-1}\mathbf{W}, F\,\mathbf{Cof}, G^{-1}\mathbf{Fib})$ is a model category, which is α-presentable by cofibrations. Furhtermore, (F,G) becomes a Quillen equivalence.

  Let us apply this to:

$$(a_{t_\emptyset}, incl) : \mathbf{PreShv}(\mathrm{Sm}/X,\mathfrak{M}) \to \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})$$

  with the $\mathbb{A}^1$-local structures on $\mathbf{PreShv}(\mathrm{Sm}/X,\mathfrak{M})$ in Def.4.5.12, p.537.
- The above left Quillen equivalence induces another left Quillen equialence:

$$a_{t_\emptyset} : \mathbf{Spect}^{\Sigma}_{T_X}(\mathbf{PreShv}(\mathrm{Sm}/X,\mathfrak{M})) \to \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M}))$$

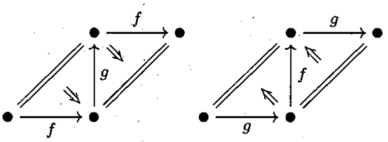
4 $\mathbb{SH}_{\mathfrak{M}}$: Dia Sch $\rightarrow$ $\mathfrak{MonoTR}$ is a stable homotopy algebraic derivataeur

4.1 2-functors, exchange structures, Voevodsky's cross functors

[Ayoub, Def.1.1.1, Rem.1.1.2, Rem.1.1.3, p.6]

Let $\mathcal{D}$ be a 2-category. e.g. $\mathcal{D} = \mathfrak{Cat}$, the 2-category of small categories.

- Let $f : X \rightarrow Y$ be a 1-morphism of $\mathcal{D}$. A right adjoint of f is the data given by:
  - A 1-morphism $g : Y \rightarrow X$,
  - Two 2-morphisms
  $$1 \xrightarrow{\eta} g \circ f \quad f \circ g \xrightarrow{\delta} 1$$
  such that the composition of each of the following two planer diagrams is the identity:



  - The two 2-morphisms η and δ are respectiely called the unit and the counit of the adjunction. Sometimes, we do not mention 2-morphisms η and δ: we simply say

    "g is a right adjoint of f ", or
    "f is a left adjoint of g ".

- The condition imposed on η and δ in the definition above is nothing but the (usual) commutatiity of the triangles :

$$\begin{tikzcd} f \arrow[r, "\delta"] \arrow[dr, equal] & f \circ g \circ f \arrow[d, "\eta"] \\ & f \end{tikzcd} \quad \begin{tikzcd} g \arrow[r, "\eta"] \arrow[dr, equal] & g \circ f \circ g \arrow[d, "\delta"] \\ & g \end{tikzcd}$$

- The duality of the 2-categories exchange the notion of the right adjunctoin and the left adjunction. More precisely, the assertions below are equivalent:
  - f is a left adjoint of g in $\mathfrak{D}$.
  - f is a right adjoint of g in $\mathfrak{D}^{1-op}$.
  - f is a right adjoint of g in $\mathfrak{D}^{2-op}$.
  - f is a left adjoint of g in $\mathfrak{D}^{1,2-op}$.
  - g is a right adjoint of f in $\mathfrak{D}$.

[Ayoub, Prop.1.1.5, p.7; Lem.1.1.6, p.8]

- Let $\mathfrak{D}$ be a 2-category and X and Y two objects of $\mathfrak{D}$. Let f and f' be two 1-morphisms and α a 2-morphism of $\mathfrak{D}$:

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & Y \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

We suppose given right adjoints (g, η, δ) and (g', η', δ') of f and f' respectively. Then there exists a unique 2-isomorphism:

$$\begin{tikzcd} Y \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] & X \arrow[from=D, to=U, Rightarrow, "\beta"] \end{tikzcd}$$

making the following diagram of 2-morphisms commutative:

$$\begin{tikzcd} & g \circ f \arrow[dr, "\alpha"] & \\ \mathbf{1} \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & & g \circ f' \\ & g' \circ f' \arrow[ur, "\beta"'] & \end{tikzcd}$$

Furthermore, the 2-morphism β is given by the composite:

$$g' \xrightarrow{\eta} g \circ f \circ g' \xrightarrow{\alpha} g \circ f' \circ g' \xrightarrow{\delta'} g$$

and make the following diagram of 2-morphisms also commutative:

$$\begin{tikzcd} & f \circ g \arrow[dr, "\delta"] & \\ f \circ g' \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] & & \mathbf{1} \\ & f' \circ g' \arrow[ur, "\delta'"'] & \end{tikzcd}$$

- The above morphism β shall be denoted by ${}^a\alpha$.
- Let f, f' and f'' be in $\mathcal{M}or_{\mathfrak{D}}(X,Y)$, the category of 1-morphisms from X to Y and g, g' and g'' be respective right adjoints. Also, let $\alpha : f \to f'$ and $\alpha' : f' \to f''$ be 2-morphisms. Then we hae the formula:

$$^a(\alpha' \circ \alpha) = ({}^a\alpha) \circ ({}^a\alpha')$$

The following Proposition 1.1.9 generalizes the preceeding Proposition 1.1.5:

[Ayoub, Prop.1.1.9, p.9; Def.1.1.10, Prop.1.1.11, p.10; Prop.1.1.12, p.11]

- Suppose a 2-morphism in a 2-category $\mathfrak{D}$ is given:

$$\begin{tikzcd} \bullet \arrow[r, "g_2"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\swarrow\alpha"] & \bullet \arrow[d, "f_1"] \\ \bullet \arrow[r, "g_1"'] & \bullet \end{tikzcd}$$

and two right adjoints (f_1', η_1, δ_1) and (f_2', η_2, δ_2) of f_1 and f_2 respectively. Then there exist a unique 2-morphism:

$$\begin{tikzcd} \bullet \arrow[r, "g_2"] \arrow[dr, phantom, "\nwarrow\beta"] & \bullet \\ \bullet \arrow[r, "g_1"'] \arrow[u, "f_2'"] & \bullet \arrow[u, "f_1'"'] \end{tikzcd}$$

making commutative one ofthe following two diagrams:

$$\begin{tikzcd} & f_1'f_1g_2 \arrow[dr, "\alpha"] & \\ g_2 \arrow[ur, "\eta_1"] \arrow[dr, "\eta_2"'] & & f_1'g_1f_2 \\ & g_2f_2'f_2 \arrow[ur, "\beta"'] & \end{tikzcd} \qquad \begin{tikzcd} & g_1f_sf_2' \arrow[dr, "\delta_2"] & \\ f_1g_2f_2' \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & g_1 \\ & f_1f_1'g_1 \arrow[ur, "\delta_1"'] & \end{tikzcd}$$

Furthermore, this 2-morphism β is given by the composition of the planar diagram:

$$\begin{tikzcd} \bullet \arrow[r, "f_2'"] \arrow[dr, equal] \arrow[dr, phantom, shift left=2, "\swarrow\delta_2"] & \bullet \arrow[r, "g_2"] \arrow[d, "f_2"] \arrow[dr, phantom, "\swarrow\alpha"] & \bullet \arrow[d, "f_1"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=2, "\swarrow\eta_1"] & \\ & \bullet \arrow[r, "g_1"'] & \bullet \arrow[r, "f_1'"'] & \bullet \end{tikzcd}$$

and made the above two diagrams commutative.
- Lack of a better terminology, we say $\underline{\beta \text{ is obtained from } \alpha \text{ by adjunctions } (f_1, f_1') \text{ and } (f_2, f_2')}$.
- (Compatibility with the vertical compositions) Suppose a planar diagram in a 2-caetegory $\mathfrak{D}$ is given:

$$\begin{tikzcd} \bullet \arrow[r, "g_3"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\swarrow\alpha'"] & \bullet \arrow[d, "h_1"] \\ \bullet \arrow[r, "g_2"'] \arrow[d, "f_2"'] \arrow[dr, phantom, "\swarrow\alpha"] & \bullet \arrow[d, "f_1"] \\ \bullet \arrow[r, "g_1"'] & \bullet \end{tikzcd}$$

and right adjoints $(f_1', \eta_1, \delta_1), (f_2', \eta_2, \delta_2), (h_1', \eta_1', \delta_1')$ and $(h_2', \eta_2', \delta_2')$ of f_1, f_2, h_1 and h_2, respectively. We denote by β and β' the 2-morphisms obtained from α and α' using preceeding adjunctions. We provide $f_1' \circ h_1'$ and $f_2' \circ h_2'$ with the structures of right adjoints of $f_1 \circ h_1$ and $f_2 \circ h_2$.
Then the 2-morphism $\beta' \circ \beta$ is the 2-morphism obtained from adjunction of $\alpha \circ \alpha'$.
- (Compatibility with the horizontal compositions) Suppose a planar diagram in a 2-caetegory $\mathfrak{D}$ is given:

$$\begin{tikzcd} \bullet \arrow[r, "h_2"] \arrow[d, "f_3"'] \arrow[dr, phantom, "\swarrow\alpha_2"] & \bullet \arrow[r, "g_2"] \arrow[d, "f_2"] \arrow[dr, phantom, "\swarrow\alpha_1"] & \bullet \arrow[d, "f_1"] \\ \bullet \arrow[r, "h_1"'] & \bullet \arrow[r, "g_1"'] & \bullet \end{tikzcd}$$

and right adjoints $(f_1', \eta_1, \delta_1), (f_2', \eta_2, \delta_2)$ and (f_3', η_3, δ_3) of f_1, f_2 and f_3, respectively. We denote by β_1 and β_2 the 2-morphisms obtained from α_1 and α_2 by adjunction.
Then the 2-morphism $\beta_1 \circ \beta_2$ is the 2-morphism obtained from $\alpha_2 \circ \alpha_1$ by adjunctions (f_1, f_1') and (f_2, f_2').

[DV, Def.2.2, p.5; Rem.2.4, p.6]

- A 2-functor (also called: pseudo-functor) from a category $\mathcal{C}$ to a 2-category $\mathfrak{D}$ is:

(a) a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathfrak{D})$;

(b) for X, Y in $\mathcal{C}$, a map from $Hom(X,Y)$ to the set of 1-morphisms from $F(X)$ to $F(Y)$;

(c) for $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, an invertible 2-morphism

$$c(g,f) : F(gf) \to F(g)F(f)$$

it is called he composition isomorphism .

The data should satisfy:

(i) (Ayoub calls this cocycle axiom) for a triple composite hgf in $\mathcal{C}$, the diagram of isomorphisms deduced from the isomorphisms (c)

$$\begin{tikzcd}
F(hgf) \arrow[r] \arrow[d] & F(hg)F(f) \arrow[d] \\
F(h)F(gf) \arrow[r] & F(h)F(g)F(f)
\end{tikzcd}$$

is commutative.

(ii) for X in $\mathcal{C}$, $F(\mathrm{Id}_X)$ is an equivalence, that is, there exists $u : F(X) \to F(X)$ such that $u \circ F(\mathrm{Id}_X)$ and $F(\mathrm{Id}_X) \circ u$ are isomorphic to the identity of $F(X)$,

- A category is a special case of a 2-category in which the only 2-morphisms are identitities. To generalize the above concept of a 2-functor to the general notion of 2-functor between 2-categories, we must modify as follows:

(b) F is a functor from $Hom(X,Y)$ to $Hom(F(X), F(Y))$,

(c) $c(g,f)$ is assumed to be functorial in f and g.

[Ayoub, Prop.1.1.17, p.13; Def.1.1.18, Lem.1.1.19, p.15]

Let $\mathcal{C}$ be a category and $\mathfrak{D}$ a 2-category. Let $\mathsf{F} : \mathcal{C} \to \mathfrak{D}$ be a covariant 2-functor. Suppose that the for any arrow $f : X \to Y$ of $\mathcal{C}$ the 1-morphism

$$\mathsf{F}(f) : \mathsf{F}(X) \to \mathsf{F}(Y)$$

admit a right adjoint.

- Then there exist:
  1. a contravariant 2-functor $\mathsf{G} : \mathcal{C} \to \mathfrak{D}$.
  2. a couple of 2-morphisms (η_f, δ_f) for each arrow f of $\mathcal{C}$, s.t.
     - for any object X of $\mathcal{C}$, $\mathsf{F}(X) = \mathsf{G}(X)$.
     - for any arrow $f : X \to Y$, the triple $(\mathsf{G}(f), \eta_f, \delta_f)$ defines a right adjoint of $\mathsf{F}(f)$.
     - for any sequence of composable arrows of $\mathcal{C}$:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

the composition of following two planar diagrams below give the same 2-morphism:

$$\begin{tikzcd}
 & & \mathsf{F}(X) \\
 & \arrow[ur, phantom, "\Downarrow\eta_f"] & \mathsf{F}(Y) \arrow[u, "\mathsf{G}(f)"'] \\
\mathsf{F}(X) \arrow[uurr, equal] \arrow[r, "\mathsf{F}(f)"'] & \mathsf{F}(Y) \arrow[ur, equal] \arrow[r, "\mathsf{F}(g)"'] \arrow[ur, phantom, "\Downarrow\eta_g"] & \mathsf{F}(Z) \arrow[u, "\mathsf{G}(g)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathsf{F}(X) & \\
 & \arrow[ur, phantom, "\Downarrow\eta_{g\circ f}"] & \arrow[r, phantom, "\overset{c_{\mathsf{G}}}{\Rightarrow}"] & \mathsf{F}(Y) \arrow[ul, "\mathsf{G}(f)"'] \\
\mathsf{F}(X) \arrow[uurr, equal] \arrow[rr, "\mathsf{F}(g\circ f)"] \arrow[dr, "\mathsf{F}(f)"'] & \arrow[d, phantom, "\Downarrow c_{\mathsf{F}}"] & \mathsf{F}(Z) \arrow[uu, "\mathsf{G}(g\circ f)"'] \arrow[ur, "\mathsf{G}(g)"'] & \\
 & \mathsf{F}(Y) \arrow[ur, "\mathsf{F}(g)"'] & &
\end{tikzcd}$$

where c_{F} and c_{G} are connection 2-morphisms for F and G respectiely. This condition is also expressed by the commutativity of the following solid diagram:

$$\begin{tikzcd}
 & & \mathsf{F}(Z) \arrow[dll, "\mathsf{G}(g)"'] \arrow[dl] & & \\
\mathsf{F}(Y) \arrow[rrrr, equal] \arrow[dr, "\mathsf{G}(f)"'] & & & & \mathsf{F}(Y) \arrow[ull, "\mathsf{F}(g)"'] \\
 & \mathsf{F}(X) \arrow[rr, equal] & & \mathsf{F}(X) \arrow[uul] \arrow[ur, "\mathsf{F}(f)"'] &
\end{tikzcd}$$

Furthermore, these are unique up to an isomorphism. We also have analogous conditions for the counit 2-morphism $(\delta_\bullet)$.

- Under the above situation, we say that the 2-functor provided with the 2-morphisms $\eta_\bullet$ and $\delta_\bullet$ is a <u>global right adjoint</u> of the 2-functor F.
- Let $f : X \to Y$ be a 1-morphism in a 2-category $\mathfrak{D}$. We suppose given (g, η, δ), a right adjoint of f. Let Z be an object of $\mathfrak{D}$.
  1. The two functors between the categories of 1-morphisms as in Lemma 1.1.6:

$$f\circ : \mathcal{M}or_{\mathfrak{D}}(Z, X) \to \mathcal{M}or_{\mathfrak{D}}(Z, Y) \quad g\circ : \mathcal{M}or_{\mathfrak{D}}(Z, Y) \to \mathcal{M}or_{\mathfrak{D}}(Z, X)$$

  form an adjoint functor pair: $g\circ$ is right ajoint to $f\circ$.

  2. The two functors between the categories of 1-morphisms as in Lemma 1.1.6:

$$\circ g : \mathcal{M}or_{\mathfrak{D}}(X, Z) \to \mathcal{M}or_{\mathfrak{D}}(Y, Z) \quad \circ f : \mathcal{M}or_{\mathfrak{D}}(Y, Z) \to \mathcal{M}or_{\mathfrak{D}}(X, Z)$$

  form an adjoint functor pair: $g\circ$ is right ajoint to $f\circ$.

[Ayoub, p.20; Def.1.2.1, p.20 (very economicak, possibly confusing, notaions!)]

- $\mathcal{C}_1$ and $\mathcal{C}_2$ are two categories such that $\mathrm{Ob}(\mathcal{C}_1) = \mathrm{Ob}(\mathcal{C}_2)$.
- Call mixed square a diagram:

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

  such that:
  - X, X', Y, Y' are objects of $\mathcal{C}_1$, so also of $\mathcal{C}_2$.
  - g, g' are arrows of $\mathcal{C}_1$.
  - f, f' are arrows of $\mathcal{C}_2$.
- $\mathcal{E}$ is a (fixed) class of mixed squares which are stable by horizontal and vertical compositions.
- $\mathfrak{D}$ is a given 2-category.
- Gien two functors $\mathsf{F}_i : \mathcal{C}_i :\to \mathfrak{D}$ $(i = 1, 2)$ such that $\mathsf{F}_1(X) = \mathsf{F}_2(X) = \mathsf{F}(X)$ for any X in $\mathrm{Ob}(\mathcal{C}_1) = \mathrm{Ob}(\mathcal{C}_2)$.

Then an exchange structure on the pair $(\mathsf{F}_1, \mathsf{F}_2)$ with respect to $\mathcal{E}$ is the data for any mixed square (C) in $\mathcal{E}$:

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

of a 2-morphism $e(C)$ of $\mathfrak{D}$ (called exchange 2-morphism associated to the mixed square (C)):

$$\begin{tikzcd} \mathsf{F}(Y') \arrow[rr, leftrightarrow, "\mathsf{F}_1(g')"] \arrow[dd, leftrightarrow, "\mathsf{F}_2(f')"'] \arrow[dr, Rightarrow] & & \mathsf{F}(X') \arrow[dd, leftrightarrow, "\mathsf{F}_2(f)"] \arrow[dl, Rightarrow] \\ & {} & \\ \mathsf{F}(Y) \arrow[rr, leftrightarrow, "\mathsf{F}_1(g)"] \arrow[ur, Rightarrow] & & \mathsf{F}(X) \arrow[ul, Rightarrow] \end{tikzcd}$$

The 2-morphism $e(.)$ means the constant (i.e. independent of mixed square). The family of these 2-morphisms must verify the following two compatible conditions:

- (compatibility with the horizontal composition of mixed squares) For any horizonal composable mixed squares C_1 and C_2:

$$\begin{tikzcd} \bullet \arrow[r, "g'"] \arrow[d, "f'"] & \bullet \arrow[r, "h'"] \arrow[d, "f'"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"] & \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

  the following solid diagram is commutative:

$$\begin{tikzcd} \bullet \arrow[rrrr, leftrightarrow, "\mathsf{F}_1(h' \circ g')"'] \arrow[ddd, leftrightarrow, "\mathsf{F}_2(f'')"'] \arrow[drr, leftrightarrow, "\mathsf{F}_1(g')"'] & & & & \bullet \arrow[ddd, leftrightarrow, "\mathsf{F}_2(f)"] \arrow[dll, leftrightarrow, "\mathsf{F}_1(h')"] \\ & & \bullet \arrow[ddd, leftrightarrow, "\mathsf{F}_2(f')"] & & \\ & & & & \\ \bullet \arrow[rrrr, leftrightarrow, "\mathsf{F}_1(h \circ g)"] \arrow[drr, leftrightarrow, "\mathsf{F}_1(g)"'] & & & & \bullet \arrow[dll, leftrightarrow, "\mathsf{F}_1(h)"] \\ & & \bullet & & \end{tikzcd}$$

- (compatibility with the vertical composition of mixed squares) For any vertical composable mixed squares C_1 and C_2:

$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "f'"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g'"] \arrow[d, "e'"'] & \bullet \arrow[d, "e"] \\ \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

  the following solid diagram is commutative:

$$\begin{tikzcd} & \bullet \arrow[rr, leftrightarrow, "\mathsf{F}_1(g'')"] \arrow[dd, leftrightarrow, "\mathsf{F}_2(e' \circ f')"] & & \bullet \arrow[dl, leftrightarrow, "\mathsf{F}_2(f)"'] \arrow[dd, leftrightarrow, "\mathsf{F}_2(e \circ f)"] \\ \bullet \arrow[ur, leftrightarrow, "\mathsf{F}_2(f')"] \arrow[rr, leftrightarrow, "\mathsf{F}_1(g')"'] \arrow[dr, leftrightarrow, "\mathsf{F}_2(e')"'] & & \bullet \arrow[dr, leftrightarrow, "\mathsf{F}_2(e)"'] & \\ & \bullet \arrow[rr, "\mathsf{F}_1(g)"'] & & \bullet \end{tikzcd}$$

- We sometimes denote the exchange on $(\mathsf{F}_1, \mathsf{F}_2)$ by the family of is exchage 2-morphisms : $(e(C))_{C \in \mathcal{E}}$.

[Ayoub, p.24; Prop.1.2.7, p.25]

For a category $\mathcal{C}$ stable by fiber products, set $\mathcal{C}_1$ and $\mathcal{C}_2$ be one of the following four cases:

1. $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$.
2. $\mathcal{C}_1 = \mathcal{C}$ and $\mathcal{C}_2 = \mathcal{C}^{op}$.
3. $\mathcal{C}_1 = \mathcal{C}^{op}$ and $\mathcal{C}_2 = \mathcal{C}$,
4. $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}^{op}$.

Suppose the following conditions are satisfied:

- Two subcategories $\mathcal{C}^1$ and $\mathcal{C}^2$ of $\mathcal{C}$ are given, both of which contain all the isomorphisms of $\mathcal{C}$ and stable by base change .
- We suppose these subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$ generate the category $\mathcal{C}$.
- For $i, j = 1, 2$, enote by $\mathcal{C}_i^j$ the category $\mathcal{C}^j$ viewed as the sub-category of $\mathcal{C}_i$ $(\mathcal{C}_i^j \subsetneqq \mathcal{C}_i)$.
- Suppose given a 2-category $\mathfrak{D}$ and covariant 2-functor

$$\mathsf{F}_i : \mathcal{C}_i \to \mathfrak{D}$$

for eqch $j \in \{1, 2\}$. We note by ${}^jF_i$ the restriction of the 2-functor F_i to $\mathcal{C}_i^j$.

We furhter suppose:

- any arrow f of $\mathcal{C}$ factorise:

$$f = p \circ s$$

with p in $\mathcal{C}_2$ and s in $\mathcal{C}_1$. We suppose given a codirectional exchage of type $\swarrow$ on each of the pairs:

- $({}^1\mathsf{F}_1, \mathsf{F}_2)$.
- $({}^2\mathsf{F}_1, \mathsf{F}_2)$,

such that for any cube:

$$\begin{tikzcd}
\bullet \arrow[rr, "h'"] \arrow[dr, "a"] \arrow[dd, "k'"'] & & \bullet \arrow[dr, "a"] \arrow[dd, "f'"] & \\
& \bullet \arrow[rr, "h"] \arrow[dd, "k"] & & \bullet \arrow[dd, "f"] \\
\bullet \arrow[rr, "g'"] \arrow[dr, "a"'] & & \bullet \arrow[dr, "a"] & \\
& \bullet \arrow[rr, "g"'] & & \bullet
\end{tikzcd}$$

haing

- the two faces parallel to the plane of the sheet : the commutative square of $\mathcal{C}_1$ with g and h in $\mathcal{C}_1^1$ and f and k in $\mathcal{C}_1^2$.
- the four faces parallel to the plane of the sheet : the mixed squares which are cartesian in $\mathcal{C}$ and with a in $\mathcal{C}_2$.

the cube in $\mathfrak{D}$:

$$\begin{tikzcd}
\bullet \arrow[rr, "\mathsf{F}_1(h')"] \arrow[dr, "\mathsf{F}_2(a)"] \arrow[dd, "\mathsf{F}_1(k')"'] & & \bullet \arrow[dr, "\mathsf{F}_2(a)"] \arrow[dd, "\mathsf{F}_1(f')"] & \\
& \bullet \arrow[rr, "\mathsf{F}_1(h)"] \arrow[dd, "\mathsf{F}_1(k)"] & & \bullet \arrow[dd, "\mathsf{F}_1(f)"] \\
\bullet \arrow[rr, "\mathsf{F}_1(g')"] \arrow[dr, "\mathsf{F}_2(a)"'] & & \bullet \arrow[dr, "\mathsf{F}_2(a)"] & \\
& \bullet \arrow[rr, "\mathsf{F}_1(g)"'] & & \bullet
\end{tikzcd}$$

formed by taking 2-morphisms:

- on the faces parallel to the plane of the sheet : the exchange 2-isomorphisms relatie to the trivial exchange on the pair $(\mathsf{F}_1, \mathsf{F}_1)$.
- on the faces paerpendicular to the plane of the sheet and horizonals : the exchange 2-isomorphisms relatie to the exchange on the pair $({}^1\mathsf{F}_1, \mathsf{F}_2)$. 78
- on the faces paerpendicular to the plane of the sheet and verticals : the exchange 2-isomorphisms relatie to the exchange on the pair $({}^2\mathsf{F}_1, \mathsf{F}_2)$.

is commutative. Then there exist a unique exchange on $(\mathsf{F}_1, \mathsf{F}_2)$ extending the two given exchanges.

[Ayoub, Def.1.2.12, p.35]

a <u>cross functor</u> denoted by

$$(\mathsf{G}^1, \mathsf{F}_1, \mathsf{F}_2, \mathsf{G}^2) : (\mathcal{C}_1, \mathcal{C}_2) \to \mathfrak{D}$$

is characterised by

1. two covariant 2-functors $\mathsf{F}_i : \mathcal{C}_i \to \mathfrak{D}$,
2. a global left adjoint G^1 of F^1,
3. a global right adjoint G^2 of F^2,
4. an exchange on $(\mathsf{F}_1, \mathsf{F}_2)$ of type $\swarrow$,
5. an exchange on $(\mathsf{G}^1, \mathsf{G}^2)$ of type $\swarrow$,

such that for any mixed square of $\mathcal{E}$:

$$\begin{tikzcd}
\bullet \arrow[r, "g'"] \arrow[d, "f'"'] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}$$

the two 2-morphisms:

$$\begin{tikzcd}
\bullet \arrow[d, "\mathsf{F}_2(f')"'] & & \bullet \arrow[ll, "\mathsf{G}^1(g')"'] \arrow[d, "\mathsf{F}_2(f)"] \\
\bullet \arrow[urr, phantom, "a^1_2 \Nwarrow"] & & \bullet \arrow[ll, "\mathsf{G}^1(g)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d, "\mathsf{F}_2(f')"'] & & \bullet \arrow[ll, "\mathsf{G}^1(g')"'] \arrow[d, "\mathsf{F}_2(f)"] \\
\bullet \arrow[urr, phantom, "\Searrow b^1_2"] & & \bullet \arrow[ll, "\mathsf{G}^1(g)"]
\end{tikzcd}$$

are the inverse isomorphisms of one another. In an equivalent way, remarking that ${}^a(a^1_2) = b^2_1$ and ${}^a(b^1_2) = a^2_1$, the two 2-morphisms

$$\begin{tikzcd}
\bullet \arrow[rr, "\mathsf{F}_1(g')"] & & \bullet \\
\bullet \arrow[u, "\mathsf{G}^2(f')"] \arrow[rr, "\mathsf{F}_1(g)"'] \arrow[urr, phantom, "a^2_1 \Nwarrow"] & & \bullet \arrow[u, "\mathsf{G}^2(f)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\mathsf{F}_1(g')"] & & \bullet \\
\bullet \arrow[u, "\mathsf{G}^2(f')"] \arrow[rr, "\mathsf{F}_1(g)"'] \arrow[urr, phantom, "\Searrow b^2_1"] & & \bullet \arrow[u, "\mathsf{G}^2(f)"']
\end{tikzcd}$$

are the inverse isomorphisms of one other.

[Ayoub, Prop.1.2.14, p.36]

Let $\mathsf{G}^1 : \mathcal{C}_1 \to \mathfrak{D}$ be a contravariant 2-functor and $\mathsf{F}_2 : \mathcal{C}_2 \to \mathfrak{D}$ a covariant 2-functor.

- We suppose given an exchange on $(\mathsf{G}^1, \mathsf{F}_2)$ of type $\nwarrow$ which turns out to be an isoexchange.
- We also suppose that G^1 and F_2 admit each right global adjoint which we denote F_1 and G^2 respectively.

Then there exist:

- an exchange on $(\mathsf{F}_1, \mathsf{F}_2)$ obtained from the isoexchange (of type $\nwarrow$) on $(\mathsf{G}^1, \mathsf{F}_2)$ and the adjunction between F_1 and G^1.
- an exchange on $(\mathsf{G}^1, \mathsf{G}^2)$ obtained from the isoexchange inverse (of type $\searrow$) on $(\mathsf{G}^1, \mathsf{F}_2)$ and the adjunction between G^2 and F_2.

The data of $(\mathsf{G}^1, \mathsf{F}_1, \mathsf{F}_2, \mathsf{G}^2)$ as well as the adjunctions and the above exchanges define a cross functor. Furthermore, the isoexchange on $(\mathsf{F}_1, \mathsf{G}^2)$ is obtained by adjunction from that of $(\mathsf{G}^1, \mathsf{F}_2)$.

[Ayoub, Th.1.3.1, p.36]

Given Data A:

Let $\mathcal{C}$ be a category with fiber products. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two sub-categories (not necessarily full) of $\mathcal{C}$ satisfying the following conditions:

1. The isomorphisms (of $\mathcal{C}$) are in $\mathcal{C}_1$ and $\mathcal{C}_2$. Arrows of $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) are stable by pull-back along any arrow of $\mathcal{C}$.
2. Any arrow f of $\mathcal{C}$ factorizes: $f = f_2 \circ f_1$ with f_i in $\mathcal{C}_i$ for $i = 1, 2$.
3. For any arrow $B \to A$ of $\mathcal{C}$ the diagonal $B \to B \times_A B$ is in $\mathcal{C}_1$.

Given Data B:

Suppose given a 2-category $\mathcal{D}$ and two covariant 2-functors:

$$\mathsf{H}_i : \mathcal{C}_i \to \mathcal{D}$$

such that for any $X \in \mathrm{Ob}(\mathcal{C})$, $\mathsf{H}_1(X) = \mathsf{H}_2(X) = \mathsf{H}(X)$. We also suppose that a codirenctional isoexchange of type $\swarrow$ on the pair $(\mathsf{H}_1, \mathsf{H}_2)$ for the class of commuutative squares having the vertical arrows in $\mathcal{C}_2$ and horisontal arrows in $\mathcal{C}_1$ is given. More explicitly that for any commutative square (C):

$$\begin{tikzcd}
\bullet \arrow[r, "i'"] \arrow[d, "f'"'] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "i"] & \bullet
\end{tikzcd}$$

with i, i' in $\mathcal{C}_1$ and f, f' in $\mathcal{C}_2$, we have a 2-isomorphism:

$$a(C) : \mathsf{H}_2(f) \circ \mathsf{H}_1(i') \xrightarrow{\sim} \mathsf{H}_1(i) \circ \mathsf{H}_2(f') : \quad \begin{tikzcd}
\bullet \arrow[r, "\mathsf{H}_1(i')"] \arrow[d, "\mathsf{H}_2(f')"'] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d, "\mathsf{H}_2(f)"] \\
\bullet \arrow[r, "\mathsf{H}_1(i)"'] & \bullet
\end{tikzcd}$$

compatible with the connection 2-isomorphisms c_i of 2-functors H_i in the usual manner (see Definition 1.2.1 (or 1.4.2. p.56)).
We furhter suppose that for $i = \mathrm{id}$ and $i' = \mathrm{id}$ (resp. $f = \mathrm{id}$ and $f' = \mathrm{id}$) the 2-morphism $a(C)$ is the identity.

$\Longrightarrow$

Conclusion:

Thn there exists a 2-functor:

$$\mathsf{H} : \mathcal{C} \to \mathcal{D}$$

such that $\mathsf{H}(X) = \mathsf{H}_1(X) = \mathsf{H}_2(X)$ for any $X \in \mathrm{Ob}(\mathcal{C})$, as well as isomorphisms of 2-functors:

$$u_i : \mathsf{H} \circ [\mathcal{C}_i \to \mathcal{C}] \to \mathsf{H}_i$$

for $i = 1, 2$ which becomes the identity on the objects and that the exchanges on $(\mathsf{H}_1, \mathsf{H}_2)$ becomes the restriction of the trivial exchange on (H, H) by the isomorphisms u_1 and u_2. Furthermore the triplet (H, u_1, u_2) is unique up to an unique isomorphism.

4.2 A summary of (pre-)derivataer theory

[Ayoub, Def.2.1.34, p.191]

A triangulated pre-derivateur with domain **Dia** is a 2-functor (not necessarily strict) 1-contravariant and 2-contravariant:
$$\mathbb{D} : \mathbf{Dia} \to \mathfrak{TR}$$
If there is no possibility of confusion, we denote for a functor u (resp. a natural transformation α) of **Dia**, u^* the functor $\mathbb{D}(u)$ (resp. α^* the natural transformation α^*).
A pre-derivateur $\mathbb{D}$ is a derivateur if it satisfies the following list of axioms.

1. $\mathbb{D}(\emptyset) = 0$, the zero category.
2. Let I and J be two categories of **Dia**. Consider the evident coupling :
$$\mathbb{D}(I \times J) \times I^{op} \to \mathbb{D}(J)$$
$$(X, i) \mapsto (i \times \mathrm{id}_J)^* X$$
where $i : \mathbf{e} \to I$. By adjunction, we deduces a functor
$$\mathbb{D}(I \times J) \to \mathsf{HOM}(I^{op}, \mathbb{D}(J))$$
This is called the I-skeleton functor . The image of $X \in \mathrm{Ob}\,(\mathbb{D}(I \times J))$ by this functor is called the I-skeleton of X. The I-skeleton functor satisfies the following three properties:
    - It is conservative for all the categories I and J of **Dia**.
    - It is full and essentially surfective for $I = \underline{1}$.
    - It is an equivalence when I is discrete.
3. For any functor $u : A \to B$ of **Dia**, the functor $u^* : \mathbb{D}(B) \to \mathbb{D}(A)$ admits a right adjoint u_* and a left adjoint $u_\sharp$.
4. Let $u : A \to B$ be a functor of **Dia** and b an object of B.
$$\begin{tikzcd} A/b \arrow[r, "j_{A/b}"] \arrow[d, "p_{A/b}"'] \arrow[dr, phantom, "\Swarrow\alpha"] & A \arrow[d, "u"] \\ \mathbf{e} \arrow[r, "b"'] & B \end{tikzcd} \qquad \begin{tikzcd} b\backslash A \arrow[r, "j_{b\backslash A}"] \arrow[d, "p_{b\backslash A}"'] \arrow[dr, phantom, "\beta\Nearrow"] & A \arrow[d, "u"] \\ \mathbf{e} \arrow[r, "b"'] & B \end{tikzcd}$$
induces by 2-functoriality the two faces of $\mathfrak{TR}$:
$$\begin{tikzcd} \mathbb{D}(A/b) \arrow[dr, phantom, "\alpha\Nearrow"] & \mathbb{D}(A) \arrow[l, "j^*_{A/b}"'] \\ \mathbb{D}(\mathbf{e}) \arrow[u, "p^*_{A/b}"] & \mathbb{D}(B) \arrow[l, "b^*"] \arrow[u, "u^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(b\backslash A) \arrow[dr, phantom, "\Swarrow\beta^*"] & \mathbb{D}(A) \arrow[l, "j^*_{b\backslash A}"'] \\ \mathbb{D}(\mathbf{e}) \arrow[u, "p^*_{b\backslash A}"] & \mathbb{D}(B) \arrow[l, "b^*"] \arrow[u, "u^*"'] \end{tikzcd}$$
The two faces obtained by adjunction:
$$\begin{tikzcd} \mathbb{D}(A/b) \arrow[d, "(p_{A/b})_*"'] \arrow[dr, phantom, "Ex^*_*\Nwarrow"] & \mathbb{D}(A) \arrow[l, "j^*_{A/b}"'] \arrow[d, "u_*"] \\ \mathbb{D}(\mathbf{e}) & \mathbb{D}(B) \arrow[l, "b^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(b\backslash A) \arrow[d, "(p_{b\backslash A})_\sharp"'] \arrow[dr, phantom, "\Searrow Ex^*_\sharp"] & \mathbb{D}(A) \arrow[l, "j^*_{b\backslash A}"'] \arrow[d, "u_\sharp"] \\ \mathbb{D}(\mathbf{e}) & \mathbb{D}(B) \arrow[l, "b^*"] \end{tikzcd}$$
are invertible.
5. Denote by $\square$ the category $\underline{1} \times \underline{1}$:
$$\begin{tikzcd} (1,1) & (0,1) \arrow[l] \\ (1,0) \arrow[u] & (0,0) \arrow[l] \arrow[u] \end{tikzcd}$$
Denote by $i_\ulcorner : \ulcorner \to \square$ the full subcategory having objects $\mathrm{Ob}(\square)\backslash\{(0,0)\}$. Denote by $i_\lrcorner : \lrcorner \to \square$ the full subcategory having objects $\mathrm{Ob}(\square)\backslash\{(1,1)\}$. Let I be a category of **Dia**. An object X of $\mathbb{D}(\square \times I)$ is called cartesian (resp. cocartesian) if the evident morphism $X \to (i_\lrcorner)_*(i_\lrcorner)^*X$ (resp. $(i_\ulcorner)_\sharp(i_\ulcorner)^*X \to X$) is an isomorphism. In a triangulated derivateur $\mathbb{D}$, an object of $\mathbb{D}(\square \times I)$ is cartesian iff. it is cocartesian.
6. Kepping the notations of the preceeding axioms. Define a functor $\Sigma : \mathbb{D}(I) \to \mathbb{D}(I)$ by the formula:
$$\Sigma = (0,0)^*(i_\ulcorner)_\sharp(i_\ulcorner)^*(1,1)_*$$
There exists an isomorphism of functors $\Sigma \to [+1]$ between Σ and the autoequivalence of suspension of the category $\mathbb{D}(I)$. Furthermore, let X be an object both cartesian and cocartesian of $\mathbb{D}(\square \times I)$. We define a morphism $(0,0)^*X \to \Sigma(1,1)^*X$ by the following composite:
$$(0,0)^*X \xleftarrow{\sim} (0,0)^*(i_\ulcorner)_\sharp(i_\ulcorner)^*X \to (0,0)^*(i_\ulcorner)_\sharp(i_\ulcorner)^*(1,1)_*(1,1)^*X = \Sigma(1,1)^*X \simeq (1,1)^*X[+1]$$
Suppose that the object $(1,0)^*A$ is null, then the triangle:
$$(1,1)^*X \to (0,1)^*X \to (0,0)^*X \to (1,1)^*X[+1]$$
is a distinguished triangle.

[Ayoub, Def.2.1.119, p.230; Def.2.1.148, p.239; Def.2.1.149, p.240]

- Let $(\mathcal{C}, \otimes)$ be a monoidal category. We say it is left closed if for any object A of $\mathcal{C}$, the functor $A \otimes -$ admits a right adjoint. We say that $\mathcal{C}$ is right closed if for any object A of $\mathcal{C}$, the functor $- \otimes A$ admits a right adjoint.
- A monoidal category $\mathcal{C}$ is right closed if the $\otimes$-opposed category $\mathcal{C}^0$ is left closed and vice versa. So just to study one type of the closed monoidal categories.

  Thereafter, we consider mainly the right closed monnoidal categories. We denote by $\underline{\mathrm{Hom}}(A, -)$ the right adjoint of $- \otimes A$. There is thus isomorphisms:

$$\hom_{\mathcal{C}}(U \otimes A, V) \xrightarrow{\sim} \hom_{\mathcal{C}}(U, \underline{\mathrm{Hom}}(A, V))$$

  as well as the arrows:

$$ev : \underline{\mathrm{Hom}}(A, v) \otimes A \to V \quad \text{and} \quad \delta : U \to \underline{\mathrm{Hom}}(A, U \otimes A)$$

  natural in U and V of $\mathcal{C}$.
- When we will need to cisider right and left closed monoidal categories, we denote, to distinguish, $\underline{\mathrm{Hom}}_g(A, -)$ and $\underline{\mathrm{Hom}}_d(A, -)$ the respective right adjoints of $A \otimes -$ and $- \otimes A$.
- A monoidal (resp. symmetric monoidal) triangulated category is is an additive monoidal category $(\mathcal{T}, \otimes, \sigma)$ (resp. $(\mathcal{T}, \otimes, \sigma, \tau)$), with a structure of triangulated category on $\mathcal{T}$ as well as the isomorphisms:

$$A[+1] \otimes B \underset{\sim}{\xrightarrow{s_g}} (A \otimes B)[+1] \underset{\sim}{\xleftarrow{s_d}} A \otimes B[+1]$$

  which are natural on $(A, B) \in \mathrm{Ob}(\mathcal{T})^2$ and commute in the evident manner with the associativity (resp. the associativity and the commutativity) isomorphisms. Also, two supplementary axioms are imposed:
  - For any distinguished triangle $A \to B \to C \to A[+1]$ and any object D of $\mathcal{T}$ the two diagrams below:

$$A \otimes D \to B \otimes D \to C \otimes D \to (A \otimes D)[+1]$$
$$D \otimes A \to D \otimes B \to D \otimes C \to (D \otimes A)[+1]$$

    are distinguished. In other words, the functors $- \otimes D$ and $D \otimes -$ provided with the isomorphisms s_g and s_d respectively, are triangulated functors.
  - For any A and B of $\mathcal{T}$, the square below is commuative up to the multiplication by -1:

$$\begin{tikzcd}
A[+1] \otimes B[+1] \arrow[r] \arrow[d] & (A[+1] \otimes B)\,[+1] \arrow[d] \\
(A \otimes B[+1])\,[+1] \arrow[r] & (A \times B)[+2]
\end{tikzcd}$$

- Let $(\mathcal{T}, \otimes)$ and $(\mathcal{T}', \otimes')$ be two triangulated monoidal (resp. symmetric monoidal) categories. A pseudo-monoidal (resp. symmetric pseudo-monoidal) triangulated functor from $\mathcal{T}$ to $\mathcal{T}'$ is a pseudo-monoidal (resp. symmetric pseudo-monoidal) functor between underlying additive monoidal categories, which is triangulated and compatible with the isomorphisms s_g and s_d.

  Suppose unit objects are given in $\mathcal{T}$ and $\mathcal{T}'$. A triangulated pseudo-monoidal and pseudo-unital functor is simply a triangulated pseudo-monoidal functor provided with an arrow e which makes it also a pseudo-monoidal and pseudo-unital.
- A monoidal (resp. symmetric monoidal) triangulated derivateur is a triangulated derivateur $\mathbb{D}$ provided with the following supplementary data:
  - For each $I \in \mathrm{Ob}(\mathbf{Dia})$ a monoidal (resp. symmetric monoidal) category structure $(\mathbb{D}(I), \otimes_I, \sigma)$.
  - For each functor $u : A \to B$ of $\mathbf{Dia}$ a monoidal (resp. symmetric monoidal) functor structure on u^*.
- A triangulated unital monoidal (resp. symmetric monoidal) derivateur is a triangulated monoidal (resp. symmetric monoidal) derivateur provided with a unit object $\mathbf{1}_I \in \mathrm{Ob}(\mathbb{D}(I))$ for each $I \in \mathrm{Ob}(\mathbf{Dia})$ and an isomorphism $u^*\mathbf{1}_I \simeq \mathbf{1}_J$ for each $u : J \to I \in \mathsf{Fl}(\mathbf{Dia})$ making u^* a unital monoidal (resp. symmetric monoidal) functor.

[Ayoub, Def.2.4.12, p.312]

Let $\mathfrak{D}$ be a strict 2-category. algebraic pre-derivateur valued in $\mathcal{D}$ is a (not necessarily strict) 2-functor $\mathbb{D}$ from the 2-category **Dia Sch** $/S$ to $\mathfrak{D}$, which is 1-contravariant and 2-contravariant. Explicitly, a pre-derivateur $\mathbb{D}$ is a set of the following data:

- An object $\mathbb{D}(\mathcal{F}\mathcal{I})$ of $\mathfrak{D}$ for any diagram of S-schemes $(\mathcal{F}, \mathcal{I})$.
- A 1-morphism $(f,\alpha)^* : \mathbb{D}(\mathcal{G},\mathcal{J}) \to \mathbb{D}(\mathcal{F},\mathcal{I})$ in $\mathbb{D}$ for any 1-morphism of diagrams of S-schemes $(f,\alpha) : (\mathcal{F},\mathcal{I}) \to (\mathcal{G},\mathcal{J})$.
- To a 2-morphism of diagrams of S-schemes:

$$\begin{tikzcd} (\mathcal{F},\mathcal{I}) \arrow[rr, bend left, "{(f,\alpha)}"] \arrow[rr, bend right, "{(f',\alpha')}"'] \arrow[rr, phantom, "\Downarrow t"] & & (\mathcal{G},\mathcal{J}) \end{tikzcd}$$

  a 2-morphism in $\mathfrak{D}$:

$$\begin{tikzcd} \mathbb{D}(\mathcal{F},\mathcal{I}) \arrow[rr, phantom, "\Uparrow t^*"] & & \mathbb{D}(\mathcal{G},\mathcal{J}) \arrow[ll, bend right, "{(f',\alpha')^*}"'] \arrow[ll, bend left, "{(f,\alpha)^*}"] \end{tikzcd}$$

- To a composable sequence of 1-morphisms of diagrams of S-schemes:

$$(\mathcal{F},\mathcal{I}) \xrightarrow{(f,\alpha)} (\mathcal{G},\mathcal{J}) \xrightarrow{(g,\beta)} (Hcal,\mathcal{K})$$

  a 2-isomorphism of connection

$$c((f,\alpha),(g,\beta)) : (f,\alpha)^* \circ (g,\beta)^* \xrightarrow{\sim} (g \circ f, \beta \circ \alpha)^*$$

  in $\mathfrak{D}$.
- These data should satisfy the properties of 1-opposite and 2-opposite of the definition 2.1.32.

[Ayoub, Def.2.4.13, p.313]

An algebraic pre-derivateur

$$\mathbb{D} : \mathbf{Dia\,Sch}\,/S \to \mathfrak{TR},$$

valued in the 2-category of triangluated categories, is called a stable homotopy algebraic derivateur when the following axioms **DerAlg0, DerAlg1, DerAlg2d, DerAlg2g, DerAlg3d, DerAlg3g, DerAlg4, DerAlg5** are satisfied:

DerAlg0 Let $(\mathcal{F},\mathcal{I})$ be a diagram of quasi-prpjective S-schemes. If $\mathcal{I}$ is a discrete category, then the 1-morphisms $i : (\mathcal{F}(i), \mathbf{e}) \to (\mathcal{F},\mathcal{I})$ for $i \in \mathrm{Ob}(\mathcal{I})$ induce an equivalence of categories:

$$\mathbb{D}(\mathcal{F},\mathcal{I}) \xrightarrow{\prod_{i\in \mathrm{Ob}(\mathcal{I})} i^*} \prod_{i\in\mathrm{Ob}(\mathcal{I})} \mathbb{D}(\mathcal{F}(i))$$

DerAlg1 Let $(\mathcal{F},\mathcal{I})$ be a diagram of quasi-projective S-schemes and $\alpha : \mathcal{J} \to \mathcal{I}$ an essentially surjective functor. Then the triangulated functor

$$\alpha^* : \mathbb{D}(\mathcal{F},\mathcal{I}) \to \mathbb{D}(\mathcal{F}\circ\alpha,\mathcal{J})$$

is conservative (i.e. detects the isomorphisms).

DerAlg2d For any 1-morphism $(f,\alpha) : (\mathcal{F},\mathcal{I}) \to (\mathcal{G},\mathcal{J})$ of $\mathbf{Dia\,Sch}\,/S$, the functor $(f,\alpha)^*$ admits a right adjoint $(f,\alpha)_*$.

DerAlg2g For any 1-morphism $(f,\alpha) : (\mathcal{F},\mathcal{I}) \to (\mathcal{G},\mathcal{J})$ of $\mathbf{Dia\,Sch}\,/S$, which is smooth argument by argument, the functor $(f,\alpha)^*$ admits a left adjoint $(f,\alpha)_\sharp$.

Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism of $\mathcal{I}$-diagrams of quasi-projective S-schemes and $\alpha : \mathcal{J} \to \mathcal{I}$ a functor in **Dia**. We have a square:

$$\begin{tikzcd}
(\mathcal{G}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"] \arrow[d, "f|_{\mathcal{J}}"'] & (\mathcal{G},\mathcal{J}) \arrow[d, "f"] \\
(\mathcal{F}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"] & (\mathcal{F},\mathcal{I})
\end{tikzcd}$$

which is commutative (even cartesian) in $\mathbf{Dia\,Sch}\,/S$.

DerAlg3d The exchange 2-morphism $\alpha^* f_* \to (f|_{\mathcal{J}})_* \circ \alpha^*$, associated to the above commutative square, is a 2-isomorphism.

DerAlg3g Suppose f is cartesian and smooth argument by argument. Then the exchange 2-morphism $(f|_{\mathcal{J}})_\sharp \circ \alpha^* \to \alpha^* f_\sharp$ is a 2-isomorphism.

DerAlg4 For any quasi-projective S-scheme X, the 2-functor:

$$\begin{aligned} \mathbb{D}(X,-) : \mathbf{Dia} &\to \mathfrak{TR} \\ \mathcal{I} &\mapsto \mathbb{D}(X,\mathcal{I}) \end{aligned}$$

is a triangulated derivateur in the sense of Definition 2.1.34.

DerAlg5 The 2-functor

$$\begin{aligned} \mathbb{D}(-,\mathbf{e}) : \mathbf{Sch}\,/S &\to \mathfrak{TR} \\ X(\text{quasi-projective}) &\mapsto \mathbb{D}(X,\mathbf{e}) \end{aligned}$$

is a stable homotopy 2-functor.

4.3 The axioms DerAlg 0 - DreAlg 4 and the Projection Formula in DerAlg 5 of $\mathbf{Dia\,Sch}\,/S \ni (\mathcal{F},\mathcal{I}) \to \mathbb{SH}^{T}_{\mathcal{M}}(\mathcal{F},\mathcal{I}) \in \mathfrak{TR}$

[Ayoub, Th. 4.5.24, p.540]

The associations $(\mathcal{F},\mathcal{I}) \mapsto \mathbb{SH}^{T}_{\mathfrak{M}}(\mathcal{F},\mathcal{I})$ and $(f,\alpha) \mapsto \mathsf{L}(f,\alpha)^*$ naturally extends to a contravariant 2-functor:

$$\mathbb{SH}^{T}_{\mathfrak{M}} : \mathbf{Dia\,Sch} \to \mathfrak{MonoTR}$$

Proof. This is shown in the following three steps:

$\mathbb{SH}^{T}_{\mathfrak{M}}(\mathcal{F},\mathcal{I})$ is a symmetric monoidal triangulated category This follows from Propositions 4.2.76, 4.2.82, 4.4.63, 4.3.77, Theorems 4.3.76, 4.1.49, and Lemma 4.1.58.

The functor $\mathsf{L}(f,\alpha)^*$ is monoidal triangular This follows from Theorem 4.5.23, Proposition 4.5.16, and Lemma 4.1.51.

Construction of the required 2-functor Then simply take the composite of 2-functors:

$$\begin{tikzcd}
\mathbf{Dia\,Sch}\,/S \arrow[r, "\mathbb{M}_T(-,-)"] \arrow[rr, bend left, "\mathbb{SH}^{T}_{\mathfrak{M}}"] & \mathfrak{ModCat} \arrow[r, "\mathrm{Ho}(-)\text{,}\mathsf{L}"] & \mathfrak{Cat} \\
(\mathcal{F},\mathcal{I}) \arrow[r, mapsto] & \mathbb{M}_T(\mathcal{F},\mathcal{I}) \arrow[r, mapsto] & \mathbb{SH}^{T}_{\mathfrak{M}}(\mathcal{F},\mathcal{I})
\end{tikzcd}$$

where $\mathfrak{ModCat}$ denotes the 2-category of model categories and left Quillen functors.

□

[Ayoub, Cor. 4.5.26, p.541]

Let $\alpha : \mathcal{J} \to \mathcal{I}$ be a functor between small categories. Let $(\mathcal{F},\mathcal{I})$ a diagram of S-schemes. The functor

$$\alpha^* : \mathbb{M}_T(\mathcal{F},\mathcal{I}) \to \mathbb{M}_T(\mathcal{F}\circ\alpha,\mathcal{J})$$

preserve the stable $\mathbb{A}^1$-equivalences. It therefore derives trivially.

[Ayoub, **DerAlg 0,1,2,3** for $\mathbb{SH}^T_{\mathfrak{M}}$ Lem.4.5.25, Cor.4.5.26, Prop.4.5.27, Lem.4.5.28, p.541; Th. 4.5.30, p.542]

The axioms **DerAlg 0, DerAlg 1, DerAlg 2, DerAlg 3, DerAlg 4** (in Definition 2.4.13) are satisfied for $\mathbb{SH}$ (in Theorem 4.5.24:

$$\begin{aligned} \mathbb{SH}^T_{\mathfrak{M}} : \mathbf{Dia\,Sch} &\to \mathfrak{MonoTR} \\ (\mathcal{F},\mathcal{I}) &\mapsto \mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F},\mathcal{I}) \end{aligned}$$

Proof.

DerAlg 0 This is trivial:

DerAlg 0 for $\mathbb{D} = \mathbb{SH}^T_{\mathfrak{M}}$

Let $(\mathcal{F},\mathcal{I})$ be a diagram of quasi-prpjective S-schemes. If $\mathcal{I}$ is a discrete category, then the 1-morphisms $i : (\mathcal{F}(i), \mathbf{e}) \to (\mathcal{F},\mathcal{I})$ for $i \in \mathrm{Ob}(\mathcal{I})$ induce an equivalence of categories:

$$\mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F},\mathcal{I}) \xrightarrow{\prod_{i\in \mathrm{Ob}(\mathcal{I})} i^*} \prod_{i\in\mathrm{Ob}(\mathcal{I})} \mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F}(i))$$

DerAlg 1

DerAlg 1 for $\mathbb{D} = \mathbb{SH}^T_{\mathfrak{M}}$

Let $(\mathcal{F},\mathcal{I})$ be a diagram of quasi-projective S-schemes and $\alpha : \mathcal{J} \to \mathcal{I}$ an essentially surjective functor. Then the triangulated functor

$$\alpha^* : \mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F},\mathcal{I}) \to \mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F}\circ\alpha,\mathcal{J})$$

is conservative (i.e. detects the isomorphisms).

For this, Ayoub proved the following Lemma 4.5.25:

> Let $(\mathcal{F},\mathcal{I})$ be a diagram of S-schemes. The functors $i^* : \mathbb{M}_T(\mathcal{F},\mathcal{I}) \to \mathbb{M}_T(\mathcal{F}(i))$ with $i \in \mathrm{Ob}(\mathcal{I})$ preserve and detect the $\mathbb{A}^1$-weak equivalences.

From this, Ayoub immediately deduced the Corollary 4.5.26:

> Let $\alpha : \mathcal{J} \to \mathcal{I}$ be a functor between small categories. Let $(\mathcal{F},\mathcal{I})$ be a diagram of S-schemes. The functor $\alpha^* : \mathbb{M}_T(\mathcal{F},\mathcal{I}) \to \mathbb{M}_T(\mathcal{F}\circ\alpha,\mathcal{J})$ preserves the $\mathbb{A}^1$-stable equivalences. It thus derives trivially.

DerAlg 2d, DerAlg 2g Ayoub obtained this statement Proposition 4.5.27, as a direct consequence of Theorem 4.5.23.

DerAlg 2d+2g for $\mathbb{D} = \mathbb{SH}^T_{\mathfrak{M}}$ (Prop.4.5.27)

For a 1-morphism $(f,\alpha) : (\mathcal{F},\mathcal{I}) \to (\mathcal{G},\mathcal{J})$ of $\mathbf{Dia\,Sch}/S$:

- The functor $\mathsf{L}(f,\alpha)^* : \mathbb{SH}(\mathcal{F},\mathcal{I}) \to \mathbb{SH}(\mathcal{G},\mathcal{J})$ admits a right adjoint $\mathsf{R}(f,\alpha)_*$.
- When $f(j)$ is smooth for any $j \in \mathrm{Ob}(\mathcal{J})$, the same functor $\mathsf{L}(f,\alpha)^* : \mathbb{SH}(\mathcal{F},\mathcal{I}) \to \mathbb{SH}(\mathcal{G},\mathcal{J})$ admits a left adjoint $\mathsf{L}(f,\alpha)_\sharp$.

DerAlg 3d, DerAlg 3g Ayoub proved this Lemma 4.5.28 by resorting to Theorem 4.5.23, which allowed to obtain the claim as a direct consequence of Lemma 4.5.7:

DerAlg 3d+3g for $\mathbb{D} = \mathbb{SH}^T_{\mathfrak{M}}$ (Lem.4.5.28)

Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism of $\mathcal{I}$-diagrams of quasi-projective S-schemes and $\alpha : \mathcal{J} \to \mathcal{I}$ a functor in **Dia**. We have a square:

$$\begin{array}{ccc} (\mathcal{G}\circ\alpha,\mathcal{J}) & \xrightarrow{\alpha} & (\mathcal{G},\mathcal{J}) \\ {\scriptstyle f|_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle f} \\ (\mathcal{F}\circ\alpha,\mathcal{J}) & \xrightarrow{\alpha} & (\mathcal{F},\mathcal{I}) \end{array}$$

which is commutative (even cartesian) in $\mathbf{Dia\,Sch}/S$. Then,

- The transformation $\alpha^*\mathsf{R}f_* \to \mathsf{R}(f|_{\mathcal{J}})_*\alpha^*$, is invertible.
- If furthermore that f is cartesian and levelwise smooth. Then $\mathsf{L}(f|_{\mathcal{J}})_\sharp\circ\alpha^* \to \alpha^*\mathsf{L}f_\sharp$ is invertible.

(to be continued...) □

[Ayoub, Prop.4.5.4, p.533; Lem.4.5.5, Lem.4.5.6, Lem.4.5.7, p.534]

- Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schemes.
  - The functor $(f,\alpha)^*$ admits a right adjoint $(f,\alpha)_*$.
  - If (f,α) is smooth argument by argument, the functor $(f,\alpha)^*$ admits a left adjoint $(f,\alpha)_\sharp$.

Proof. Use the decomposition $(f,\alpha) = \alpha \circ f$:

The case of $f : (\mathcal{G},\mathcal{J}) \to (\mathcal{F}\circ\alpha,\mathcal{J})$ f^* is the inverse image w.r.t. the functor

$$f = (- \times_{\mathcal{F}} \mathcal{G}) : \mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}) \to \mathrm{Sm}/(\mathcal{G},\mathcal{J})$$

It admits a right adjoint, i.e. the direct image functor which to a presheaf H on $\mathrm{Sm}/(\mathcal{G},\mathcal{J})$ associates $H \circ f$.

When the morphism $f(j)$ are smooth, the functor $f : \mathrm{Sm}/(\mathcal{F}\circ\alpha,\mathcal{J}) \to \mathrm{Sm}/(\mathcal{G},\mathcal{J})$ admits a right adjoint c_f which to $(V \to \mathcal{G}(j), j)$ associates the pair $(V \to \mathcal{G}(j) \to \mathcal{F}(\alpha(j),j))$. The left adjoint $f_\sharp$ of f^* is then given by c_f^*. Then furhtermore $f^* = (c_f)_*$.

The case of $\alpha : (\mathcal{F}\circ\alpha,\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ The functor α^* is constructed as the direct image of the functor $\overline{\alpha}$. It thus admits a left adjoint $\alpha_\sharp = \overline{\alpha}^*$ and the right adjoint $\alpha_* = \overline{\alpha}^!$. □

- Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-morphisms. For $i \in \mathrm{Ob}(\mathcal{I})$ we form the square boundary in **Dia Sch** $/S$:

$$\begin{tikzcd} (\mathcal{G}/i,\mathcal{J}/i) \arrow[r, "(\mathrm{id},u_i)"] \arrow[d, "(f/i)"'] \arrow[dr, phantom, "\not\Downarrow_r"] & (\mathcal{G},\mathcal{J}) \arrow[d, "(f,\alpha)"] \\ \mathcal{F}(i) \arrow[r, "(\mathrm{id}_{\mathcal{F}(i)},i)"'] & (\mathcal{F},\mathcal{I}) \end{tikzcd}$$

Then the natural transformation

$$(\mathrm{id}_{\mathcal{F}(i)},i)^*(f,\alpha)_* \to (f/i)_*(\mathrm{id},u_i)^*$$

is invertible.

- When $\mathcal{F}$ and $\mathcal{G}$ are constant valued at the S-scheme F, we can form the square face:

$$\begin{tikzcd} (F,i\backslash\mathcal{J}) \arrow[r, "u_i"] \arrow[d, "\alpha/i"'] \arrow[dr, phantom, "{}^r\!\not\Uparrow"] & (F,\mathcal{J}) \arrow[d, "\alpha"] \\ (F,i) \arrow[r, "i"] & (F,\mathcal{I}) \end{tikzcd}$$

Then the natural transformation
$$(\alpha/i)_\sharp u_i^* \to i^*\alpha_\sharp$$
is invertible.

- Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism of $\mathcal{I}$-diagrams of S-schemes. For a functor $\alpha : \mathcal{J} \to \mathcal{I}$, we form the commutatie square:

$$\begin{tikzcd} (\mathcal{G}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"] \arrow[d, "f_{|\mathcal{J}}"'] & (\mathcal{G},\mathcal{I}) \arrow[d, "f"] \\ (\mathcal{F}\circ\alpha,\mathcal{J}) \arrow[r, "\alpha"'] & (\mathcal{F},\mathcal{I}) \end{tikzcd}$$

Then the natural transformation
$$\alpha^* f_* :\to (f_{|\mathcal{J}})_* \alpha^*$$
is invertible. Suppose furthermore that f is cartesian and smooth level by level. Then

$$(f_{|\mathcal{J}})_\sharp \alpha^* \to \alpha^* f_\sharp$$

is invertible.

~~[Ayoub, Der~~**Alg 4** for $\mathbb{SH}^T_{\mathfrak{M}}$ p.541; p.542; Lem.4.5.29, Th. 4.5.30, p.542]

The axioms **DerAlg 0, DerAlg 1, DerAlg 2, DerAlg 3, DerAlg 4** (in Definition 2.4.13) are satisfied for $\mathbb{SH}$ (in Theorem 4.5.24:

$$\begin{aligned} \mathbb{SH}^T_{\mathfrak{M}} : \mathbf{Dia\,Sch} &\to \mathfrak{MonoTR} \\ (\mathcal{F}, \mathcal{I}) &\mapsto \mathbb{SH}^T_{\mathfrak{M}}(\mathcal{F}, \mathcal{I}) \end{aligned}$$

Proof (continued).

DerAlg 4

DerAlg 4 for $\mathbb{D} = \mathbb{SH}^T_{\mathfrak{M}}$

For any quasi-projective S-scheme X, the 2-functor:

$$\begin{aligned} \mathbb{SH}^T_{\mathfrak{M}}(X, -) : \mathbf{Dia} &\to \mathfrak{TR} \\ \mathcal{I} &\mapsto \mathbb{SH}^T_{\mathfrak{M}}(X, \mathcal{I}) \end{aligned}$$

is a triangulated derivateur in the sense of Definition 2.1.34.

In the Definition 2.1.34, only part 4, 5, 6 are verificd:

Part 4 For this, Ayoub proved the following Lemma 4.5.29:

Recall Lemma 4.5.5 and Lemma 4.5.6:

- Let $(f, \alpha) : (\mathcal{G}, \mathcal{J}) \to (\mathcal{F}, \mathcal{I})$ be a 1-morphism of diagrams of S-morphisms. For $i \in \mathrm{Ob}(\mathcal{I})$ we form the square boundary in **Dia Sch** $/S$:

$$\begin{tikzcd} (\mathcal{G}/i, \mathcal{J}/i) \arrow[r, "(\mathrm{id}, u_i)"] \arrow[d, "(f/i)"'] \arrow[dr, phantom, "\not\Downarrow_r"] & (\mathcal{G}, \mathcal{J}) \arrow[d, "(f, \alpha)"] \\ \mathcal{F}(i) \arrow[r, "(\mathrm{id}_{\mathcal{F}(i)}, i)"'] & (\mathcal{F}, \mathcal{I}) \end{tikzcd}$$

  Then the natural transformation

$$(\mathrm{id}_{\mathcal{F}(i)}, i)^* (f, \alpha)_* \to (f/i)_* (\mathrm{id}, u_i)^*$$

  is invertible.

- When $\mathcal{F}$ and $\mathcal{G}$ are constant valued at the S-scheme F, we can form the square face:

$$\begin{tikzcd} (F, i\backslash\mathcal{J}) \arrow[r, "u_i"] \arrow[d, "\alpha/i"'] \arrow[dr, phantom, "{}^r\!\not\Uparrow"] & (F, \mathcal{J}) \arrow[d, "\alpha"] \\ (F, i) \arrow[r, "i"] & (F, \mathcal{I}) \end{tikzcd}$$

  Then the natural transformation

$$(\alpha/i)_\sharp u_i^* \to i^* \alpha_\sharp$$

  is invertible.

Then, Ayoub proved in Lemma 4.5.29 that these levelwise invertiblitities hold also at the model category $\mathbb{M}_T(-,-)$ level:

$$(\mathrm{id}_{\mathcal{F}(i)}, i)^* \mathsf{R}(f, \alpha)_* \to \mathsf{R}(f/i)_* (\mathrm{id}, u_i)^*, \quad \mathsf{L}(\alpha/i)_\sharp u_i^* \to i^* \mathsf{L}\alpha_\sharp$$

are invertible.

Part 5, 6 These follow immediately from the construction of the triangulated sturucture of the homotopy category (see Theorem 4.1.49) and from Theorem 4.1.56 (homotopy cartesian = homotopy cocartesian).

□

~~[Ayoub, Pro~~jection formula in **DerAlg 5** for $\mathbb{SH}^{T}_{\mathfrak{M}}$ Prop.4.5.17, p.538: Prop.4.5.31, p.542]

The projection formulae [Ayoub, Def. 4.5.17, p.538]

1. Let $f : \mathcal{G} \to \mathcal{F}$ be a Cartesian smooth morphism of $\mathcal{I}$-diagrams of S-schemes. For $K \in \mathrm{Ob}\,(\mathbf{PreShv}\,(\mathrm{Sm}/(\mathcal{F},\mathcal{I}),\mathfrak{M}))$ and $M \in \mathrm{Ob}\,(\mathbf{PreShv}\,(\mathrm{Sm}/(\mathcal{G},\mathcal{I}),\mathfrak{M}))$, the strucural morphism of f^*-projector is invertible:
$$f_\sharp\,(f^*(K)\otimes M) \xrightarrow{\sim} K \otimes f_\sharp(M)$$
2. Let $\alpha : \mathcal{J} \to \mathcal{I}$ be a functor of **Dia** and X a S-scheme. For $K \in \mathrm{Ob}\,(\mathbf{PreShv}\,(\mathrm{Sm}/(X,\mathcal{I}),\mathfrak{M}))$ and $M \in \mathrm{Ob}\,(\mathbf{PreShv}\,(\mathrm{Sm}/(X,\mathcal{J}),\mathfrak{M}))$, the following morphism is invertible:
$$\alpha_\sharp\,(\alpha^*(K)\otimes M) \xrightarrow{\sim} K \otimes \alpha_\sharp M$$
3. Let $(f,\alpha) : (\mathcal{G},\mathcal{J}) \to (\mathcal{F},\mathcal{I})$ be a 1-morphism of diagrams of S-schems. Denote by $\pi_{\mathcal{F},\mathcal{I}}$ the structual projection:
$$\pi_{\mathcal{F},\mathcal{I}} : (\mathcal{F},\mathcal{I}) \to S.$$
For $T \in \mathrm{Ob}\,(\mathbf{PreShv}\,(\mathrm{Sm}/S,\mathfrak{M}))$, $M \in \mathrm{Ob}\,(\mathbf{PreShv}\,(\mathrm{Sm}/(\mathcal{G},\mathcal{J}),\mathfrak{M}))$, the following is invertible:
$$(f,\alpha)_\sharp\,((f,\alpha)^*\,(\pi_{\mathcal{F},\mathcal{I}}T)\otimes M) \xrightarrow{\sim} (\pi_{\mathcal{F},\mathcal{I}}T)\otimes (f,\alpha)_\sharp M.$$

An outline of the proof:

Case of (1) By Lemma 4.5.7, we may reduce to the case $\mathcal{I} = *$; we may assume $f : Y \to X$, a smooth morphism of S-schemes. Since involved functors commute with colimits, we may suppose $K = U \otimes (A_{cst})$ and $M = V \otimes (B_{cst})$ with U a smooth X-scheme, V a smooth Y-scheme, and $A, B \in \mathrm{Ob}(\mathfrak{M})$. We then have the following chain of isomorphisms whose composition is seen to coincide with the canocical morphism in the statement:
$$f_\sharp\,(f^*(U\otimes A_{cst})\otimes(V\otimes B_{cst})) \simeq f_\sharp\,(((U\times_X Y)\otimes A_{cst})\otimes(V\otimes B_{cst})) \simeq f_\sharp\,(((U\times_X Y)\times_Y V)\otimes(A\otimes B)_{cst})$$
$$\simeq (U\times_X(V\to Y\to X))\otimes(A\otimes B)_{cst} \simeq (U\otimes A_{cst})\otimes((V\to Y\to X)\otimes B_{cst}) \simeq (U\otimes A_{cst})\otimes f_\sharp(V\otimes B_{cst})$$

Case of (2) By Lemma 4.5.6, we may reduce to the case $\mathcal{I} = *$; we may assume $\alpha : \mathcal{J} \to *$. We may suppose $K = U \otimes (A_{cst})$ and $M = (V,j)\otimes(B_{cst})$ with U, V smooth X-schemes, $j \in \mathrm{Ob}(\mathcal{J})$, and $A, B \in \mathrm{Ob}(\mathfrak{M})$. We then have the following chain of isomorphisms :
$$\alpha_\sharp\,(\alpha^*(U\otimes A_{cst})\otimes((V,j)\otimes B_{cst})) \simeq \alpha_\sharp\,((U\times_X V,j)\otimes(A\otimes B)_{cst})$$
$$\simeq (U\times_X V)\otimes(A\otimes B)_{cst} \simeq (U\otimes A_{cst})\otimes\alpha_\sharp\,((V,j)\otimes B_{cst})$$

Case of (3) We may suppose $T = U\otimes(A_{cst})$ and $M = (V,j)\otimes(B_{cst})$ with U a smooth S-scheme, $j \in \mathrm{Ob}(\mathcal{J})$, V a smooth $\mathcal{G}(j)$-scheme, and $A, B \in \mathrm{Ob}(\mathfrak{M})$. We then have the following chain of isomorphisms :
$$(f,\alpha)_\sharp\,((f,\alpha)^*\pi^*_{\mathcal{F},\mathcal{I}}(U\otimes A_{cst})\otimes((V,j)\otimes B_{cst})) \simeq (f,\alpha)_\sharp\,((U\times_S V,j)\otimes(A\otimes B)_{cst})$$
$$\simeq (U\times_S V\to\mathcal{F}(\alpha(j)),j)\otimes(A\otimes B)_{cst}\pi^*_{\mathcal{F},\mathcal{I}}(U\otimes A_{cst})\otimes((V\to\mathcal{F}(\alpha(j)),j)\otimes B_{cst})$$
$$\simeq \pi^*_{\mathcal{F},\mathcal{I}}(U\otimes A_{cst})\otimes(f,\alpha)_\sharp\,((V,j)\otimes B_{cst})$$
□

Projection formula in **DerAlg 5** for $\mathbb{SH}^{T}_{\mathfrak{M}}$, (Prop.4.5.31)

1. Let $f : \mathcal{G} \to \mathcal{F}$ be a cartesian smooth morphism of $\mathcal{I}$-diagrams of S-schemes. For $K \in \mathrm{Ob}\,(\mathbb{M}_T(\mathcal{F},\mathcal{I}))$ and $M \in \mathrm{Ob}\,(\mathbb{M}_T(\mathcal{G},\mathcal{I}))$, the following morphism is invertible:
$$\mathsf{L}f_\sharp\left(\mathsf{L}f^*(K)\otimes^{\mathsf{L}} M\right) \to K\otimes^{\mathsf{L}}\mathsf{L}f_\sharp M$$
2. Let $\alpha : \mathcal{J} \to \mathcal{I}$ be a functor between small categories and X a S-scheme. For $K \in \mathrm{Ob}\,(\mathbb{M}_T(X,\mathcal{I}))$ and $M \in \mathrm{Ob}\,(\mathbb{M}_T(X,\mathcal{J}))$, the following morphism is invertible:
$$\mathsf{L}\alpha_\sharp\left(\alpha^*(K)\otimes^{\mathsf{L}} M\right) \to K\otimes^{\mathsf{L}}\mathsf{L}\alpha_\sharp M$$

An outline of the proof:

Case of (1) This follows from the above Prop.4.5.17 and the fact that all the functors involved are left Quillen w.r.t. the $\mathbb{A}^1$-stable semi-projective structure (using that f is Cartesian).

Case of (2) Apply Lem.4.5.29 to the case $\mathcal{I} = *$. In this case, we are asked to show that for K projective cofibrant the functor $\alpha^*(K)\otimes -$ is left Qullen w.r.t. the $\mathbb{A}^1$-stable projective structure, as an exercise □

4.4 A cross functor structure on $\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_\sharp, {}^{\mathrm{Liss}}\mathsf{H}^*\right)$ and the smooth base change theorem

[Ayoub, Prop.1.4.12, p.61]

(There exists a cross functor structure on the quadruplet:

$$\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_\sharp, {}^{\mathrm{Liss}}\mathsf{H}^*\right) \quad)$$

We have a cross functor from $(\mathbf{Sch}/S)$ and $(\mathbf{Sch}/S)^{\mathrm{Liss}}$ to $\mathfrak{TR}$ with respect to the class of cartesian squares with vertical smooth. This cross functor is defined by the data:

- The 2-functor H^* and its right global adjoint H_*,
- The 2-functor ${}^{\mathrm{Liss}}\mathsf{H}^*$ and its left global adjoint ${}^{\mathrm{Liss}}\mathsf{H}_\sharp$,
- The trivial exchange structure on $(\mathsf{H}^*, {}^{\mathrm{Liss}}\mathsf{H}^*)$,
- The exchange structure on $(\mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_\sharp)$ deduced from the isoexchange of type $\nwarrow$, inverse of the isoexchange on $(\mathsf{H}^*, {}^{\mathrm{Liss}}\mathsf{H}_\sharp)$ (with respect to the cartesian squares) and the global adjunction between H_* and H^*.

For a cartesian square (C) :

$$\begin{tikzcd} \bullet \arrow[r, "g'"] \arrow[d, "f'"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

with f smooth, the exchange 2-morphism of the exchange structure on $(\mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}^*)$ is given by Ex^*_* applied to the commutative square:

$$\begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

The exchange 2-isomorphism of th eexchange sructure on $\left(\mathsf{H}^*, {}^{\mathrm{Liss}}\mathsf{H}_\sharp\right)$ is given by the 2-morphism $Ex^*_\sharp(C)$. Finally, the exchange 2-morphism $Ex_{*,\sharp}(C)$ relative to the exchange on $\left(\mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_\sharp\right)$ is given by the composite:

$$f_\sharp g'_* \to g_* g^* f_\sharp g'_* \xrightarrow{(Ex^*_\sharp)^{-1}} g_* f'_\sharp g'^* g'_* \to g_* f'_\sharp$$

[Ayoub, Prop.1.4.13, p.61]

(<u>Smooth base change theorem</u>) Consider a cartesian square in $\mathbf{Sch}/S$:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with g smooth. Then the exchange 2-morphism

$$Ex^*_* : g^* f_* \xrightarrow{\sim} f'_* g'^*$$

is invertible. In other words, the exchange on $\left({}^{\mathrm{Liss}}\mathsf{H}^*, \mathsf{H}_*\right)$ is an isoexchange (with respect to the cartesian squares).

4.5 Construction of $i^!$ for a closed immersion

[Ayoub, $i^! : \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M}) \to \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z, \mathfrak{M})$ Prop.4.5.32, Lem.4.5.33, p.543; Th.4.5.34, Rem.4.5.35, p.544]

Let $i : Z \to X$ be a closed immersion of S-schemes. Then

- The functor
$$i_* : \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/Z, \mathfrak{M}) \to \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/X, \mathfrak{M})$$
is a left Quillen functor for the injective structures $(\mathbf{W}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{inj})$ (with topology $t_\emptyset$) and $(\mathbf{W}_\tau, \mathbf{Cof}_{inj}, \mathbf{Fib}_{\tau-inj})$ (with topology τ).
- Here the right adjoint is given by:
$$i^! = \mathbf{b} \circ i^!_{presheaf} \circ inc\, \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M}) \to \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z, \mathfrak{M})$$
where $\mathbf{b}$ is the right adjoint defined in p.543:
$$\mathbf{b} : \mathbf{PreShv}(\mathrm{Sm}/X, \mathcal{C}) \to \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathcal{C})$$
which associates to a presheaf F the $t_\emptyset$-sheaf $b(F)$ given by
$$(\mathbf{b}F)(U) = F(U) \times_{F(\emptyset/X)} *_{\mathcal{C}} \quad (U \in \mathrm{Ob}(\mathrm{Sm}/X))$$
and $i^!_{presheaf}$ is the usual right Kan extension:
$$\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/Z, \mathfrak{M})$$
$$H \mapsto i^!_{presheaf} H = ((V \to Z) \mapsto \mathbf{Lim}_{U \times_X Z \to V}\, H(U))$$
Now apply the Bousfield localisation w.r.t. the $\mathbb{A}^1$-equivalences to this Quillen adjunction as in Remark 4.2.59, we obation the following Quillen adjunction whose right adjoint is given by $\mathrm{R}i^!$, which we still use denote by $i^!$: On the other hand, since i_* preseres $\mathbb{A}^1$-weak equivalces, i_* derives trivially $\mathrm{L}i_* \simeq i_*$ below:
- (To prove the above claim (Prop.4.5.32) that i^* is left Quillen, by Def.44.60, Th.4.4.61, we need the following claim (Lem.4.5.33);)

  The natural transformation
$$a_\tau \circ i_* \to i_* \circ a_\tau$$
  is invertible when evaluated on the $t_\emptyset$-sheaf of sets.

  Rough outline of the proof of Lem.4.5.33.

  – For $F \in \mathbf{PreShv}(\mathrm{Sm}/Z)$ and U a smooth X-scheme, using the functor L in Proposition 4.4.8, we have $\mathrm{L}(i_*F)(U) =$
  $\mathrm{Colim}_{(U_i \to U)_i \in \mathrm{Cov}_\tau(U)} \mathrm{Eq}\left(\prod_i F(U_i \times_X Z) \rightrightarrows \prod_{i,j} F((U_i \times_U U_j) \times_X Z)\right)$

  – When $U \times_X Z \neq \emptyset$,
$$\mathrm{Cov}_\tau(U) \to \mathrm{Cov}_\tau(U \times_X Z)$$
$$(U_i \to U)_i \mapsto (U_i \times_X Z \to U \times_X Z)_i$$
  is cofinal and so $\mathrm{L}(i_*F)(U)$ is isomorphic to:
$$\mathrm{L}(i_*F)(U) = \mathrm{Colim}_{(V_j \to U \times_X Z)_i \in \mathrm{Cov}(U \times_X Z)} \mathrm{Eq}\left(F(V_i) \rightrightarrows \prod_{i,j} F(V_i \times_{U \times_X Z} V_j)\right)$$
$$= \mathrm{L}F(U \times_X Z) = i_* \mathrm{L}F(U).$$
  – On the other hand, for a $t_\emptyset$-sheaf F, when $U \times_X Z \simeq \emptyset/Z$,
$$\mathrm{L}(i_*F)(U) = * = \mathrm{L}(F)(\emptyset) = \mathrm{L}F(U \times_X Z) = i_*\mathrm{L}F(U).$$
  – Thus $\mathrm{L}(i_*F) \simeq i_*\mathrm{L}F$, which is equal to $a_\tau(i_*F) \simeq i_* a_\tau F$.

  □

- The functor
$$i_* : \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/Z, \mathfrak{M}) \to \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/X, \mathfrak{M})$$
is also a left Quillen functor for the $\mathbb{A}^1$-local injective structures $(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{\mathbb{A}^1-inj})$ (with topology τ). Actually, i_* preserves the $\mathbb{A}^1$-weak equivalences, and so derives trivially:
$$\mathrm{L}i_* \simeq i_* \simeq \mathrm{R}i_*$$

[Ayoub, $i^! : \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})\right) \to \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_Z)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M})\right)$ Prop.4.5.45, p.551; Lem.4.5.46, p.552]

Let $i : Z \to X$ be a closed immersion of S-schemes. Then the functor

$$i_* : \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_Z)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M})\right) \to \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})\right),$$

obtained by applying:

$$\begin{cases} i_* \text{ levelwise;} \\ \text{the natural transformation } a_{t_\emptyset}(T_X)\otimes i_*(-) \to i_*\left(a_{t_\emptyset}(T_Z)\otimes -\right), \text{to the structure maps,}\end{cases}$$

is a left Quillen functor with respect to the stable injective structures on the spectra induced from the $\mathbb{A}^1$-local injecture structures $\left(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{inj-\mathbb{A}^1}\right)$ on the categories of t-sheaves. (Unlike the left Quillen functor j^* for the complementary open immersion treated after Prop.4.5.43, this is NOT covered by Th.4.5.23).

Rough outline of the proof.

- By Def.4.1.24, Prof.4.1.23; Def.4.3.29; Def.4.5.12, it suffices to:
  - construct a right adjoint of i_*;
  - show i_* preserves the stable injective cofibrations and stable injective $\mathbb{A}^1$-trivial cofibrations.
- We can construct the right adjoint $i^!$ of i_* : For

  $$\mathbf{E} \in \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})\right),$$

  define $i^!\mathbf{E}$ by:

$$\begin{cases} i^!\mathbf{E} := \mathbf{F} = (\mathbf{F}_n)_{n\in\mathbb{N}} \in \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_Z)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M})\right), \\ \mathbf{F}_n := \mathsf{Eq}\left(\prod_{k\in\mathbb{N}}\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k}, i^!\mathbf{E}_{k+n}\right) \rightrightarrows \prod_{l\in\mathbb{N}}\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes l}, i^!\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_X),\mathbf{E}_{l+1+n}\right)\right)\right)\end{cases}$$

  where Σ_n operates on $\mathbf{F}_n$ by the following restriction $1\times\Sigma_n \subset \Sigma_m\times\Sigma_n \subset \Sigma_{m+n}$ of the action of Σ_{m+n} on $\mathbf{E}_{m+n}$. The two arrows in this equalizer are given by the adjoints of the assemby morphism of $a_{t_\emptyset}(T_Z)$-spectrum $\mathbf{E}$ and by the composite:

$$\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k}, i^!(-)\right) \simeq \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k-1}, \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z), i^!(-)\right)\right) \to \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k-1}, i^!\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_X),-\right)\right)$$

  We can easily verify that the product of arrows:

  $$\mathsf{F}_n \to \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k}, i^!\mathbf{E}_{k+n}\right) \simeq \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z), \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k-1}, i^!\mathbf{E}_{k-1+1+n}\right)\right)$$

  factors through the sub-object $\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z), \mathbf{F}_{n+1}\right)$.
  We thus obtain a symmetric $a_{t_\emptyset}(T_Z)$-spectrum $\mathbf{F}$, for which we can verify the funcor $\hom(i_*(-),\mathbf{E})$ is represented by $\mathbf{F}$.
- By Theorem 4.5.34, i_* levelwise preserves injective cofibrations and injective $\mathbb{A}^1$-trivial cofibrations.
- To strengthen the above preservation property to the stable setting as in Definition 4.3.29, we shoudl verify that, for any injective cofibrant $K \in \mathrm{Ob}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M})\right)$ and $p\in\mathbb{N}$, i_* sends ω^p_K of page 484 to stable $\mathbb{A}^1$-equivalences. To see this, consider:

$$\begin{array}{ccc} \mathrm{Sus}^{p+1}_{a_{t_\emptyset}(T_X),\Sigma}\left(a_{t_\emptyset}(T_X)\otimes i_*K\right) & \xrightarrow{\omega^p_{i_*K}} & \mathrm{Sus}^p_{a_{t_\emptyset}(T_X),\Sigma}(i_*K) \\ \downarrow & & \downarrow \\ i_*\,\mathrm{Sus}^{p+1}_{a_{t_\emptyset}(T_Z),\Sigma}\left(a_{t_\emptyset}(T_Z)\otimes K\right) & \xrightarrow{i_*\omega^p_K} & i_*\,\mathrm{Sus}^p_{a_{t_\emptyset}(T_Z),\Sigma}(K) \end{array}$$

  To see $i_*\omega^p_K$ is an $\mathbb{A}^1$-equivalence, for $\omega^p_{i_*K}$ is so, it suffices to show the vertical arrows are levelwise $\mathbb{A}^1$-weak equivalences. Thus, suffices to show the arrows:

  $$a_{t_\emptyset}\left(T_X^{\otimes r}\right)\otimes i_*K \to i_*\left(a_{t_\emptyset}\left(T_Z^{\otimes r}\right)\otimes K\right)$$

  are $\mathbb{A}^1$-weak equivalences.
- To see this, let j be the immersion of the complementary open, and we apply the "unstable" conservation theorem w.r.t. $(\mathsf{L}i^*, j^*)$ of Corollary 4.5.44:

  $\underline{j^*}$ $j^*\left[a_{t_\emptyset}\left(T_X^{\otimes r}\right)\otimes i_*K\right] \to j^*\left[i_*\left(a_{t_\emptyset}\left(T_Z^{\otimes r}\right)\otimes K\right)\right]$ is invertible.

  Proof. This is the unique arrow between null objects for $j^*i_* = 0$. □

  $\underline{\mathsf{L}i^*}$ $\mathsf{L}i^*\left[a_{t_\emptyset}\left(T_X^{\otimes r}\right)\otimes i_*K\right] \to \mathsf{L}i^*\left[i_*\left(a_{t_\emptyset}\left(T_Z^{\otimes r}\right)\otimes K\right)\right]$ is invertible.

  Proof. For this, consider the commutative diagram:

$$\begin{array}{ccc} \mathsf{L}i^*\left[a_{t_\emptyset}\left(T_X^{\otimes r}\right)\otimes i_*K\right] & \longrightarrow & \mathsf{L}i^*\left[i_*\left(a_{t_\emptyset}\left(T_Z^{\otimes r}\right)\otimes K\right)\right] \\ \wr\downarrow & & \downarrow \\ a_{t_\emptyset}\left(T_Z^{\otimes r}\right)\otimes \mathsf{L}i^*i_*K & \longrightarrow & a_{t_\emptyset}\left(T_Z^{\otimes r}\right)\otimes K \end{array}$$

  Then the claim follows from the latter part of Cor.4.5.44 which claims $\mathsf{L}i^*i_*$ is invertible. (Here, we need not to derive i_* for we are considering the sheaves like Prop.4.5.34, Rem.4.5.35, unlike Cor.4.5.44).

[Ayoub, 1.4.4. $i^!: \mathsf{H}(X) \to \mathsf{H}(Z)$, Lem.1.4.6, Lem.1.4.7, Lem.1.4.8, p.58; Prop.1.4.9, Cor.1.4.10, p.59]

Next, we would like to define

$$i^! : \mathsf{H}(X) \to \mathsf{H}(Z)$$

for a closed immersion $i : Z \to X$. As is expressed in [Ayoub, p.58], for each $A \in \mathrm{Ob}(\mathsf{H}(X))$, we would like to have a distinguished triangle in $\mathsf{H}(X)$:

$$i_*i^!A \to A \to j_*j^*A \xrightarrow{[+1]} i_*i^!A[+1]$$

Since $i^*i_*B \xrightarrow{\simeq} B$ for $B \in \mathrm{Ob}(\mathsf{H}(X))$, we shall define

$$i^!A- = i^*\,\mathrm{Cone}\,(A \to j_*j^*A)\,[-1]$$

Now the difficulty is making this construction functorial. This difficulty is taken care of in the following:

Let $j : U \to X$ an open immersion (between quasi-projective S-schemes) and $i : Z \to X$ be a complementary closed immersion.

- There exists a unique 2-morphism ϕ such that the sequence:

$$j_\sharp j^* \xrightarrow{\delta^*_\sharp(j)} \mathrm{Id}_{\mathsf{H}(X)} \xrightarrow{\eta^*_*(i)} i_*i^* \xrightarrow{\phi} j_\sharp j^*[+1]$$

  becomes a distinguished 2-triangle.
- The 1-morphism j^*i_* is null.
- Suppose for each $A \in \mathrm{Ob}\,(\mathsf{H}(X))$ a distinguished triangle in $\mathsf{H}(X)$ is chosen:

$$A \to j_*j^*A \xrightarrow{\theta} C(A) \to A[+1]$$

  Then for any morphism $\alpha : A \to B$ in $\mathsf{H}(X)$ there exists a unique morphism $C(\alpha) : C(A) \to C(B)$ making the following square commutative:

$$\begin{tikzcd} C(A) \arrow[r] \arrow[d, "C(\alpha)"'] & A[+1] \arrow[d, "\alpha[+1]"] \\ C(B) \arrow[r] & B[+1] \end{tikzcd}$$

  This same morphism also make the next diagram commutative:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & j_*j^*A \arrow[r, "\theta"] \arrow[d, "\alpha"'] & C(A) \arrow[r] \arrow[d, "C(\alpha)"'] & A[+1] \arrow[d, "\alpha[+1]"] \\ B \arrow[r] & j_*j^*B \arrow[r, "\theta"] & C(B) \arrow[r] & B[+1] \end{tikzcd}$$

  Then the associations: $A \to C(A)$ and $\alpha \to C(\alpha)$ define a triangulated endofunctor of $\mathsf{H}(X)$.
- 1. For $i : Z \to X$ there exists a 1-morphism

$$i^! : \mathsf{H}(X) \to \mathsf{H}(Z)$$

  and a distinguished 2-triangle:

$$i_*i^! \xrightarrow{\delta^!_*} \mathrm{Id}_{\mathsf{H}(X)} \xrightarrow{\eta^*_*} j_*j^* \xrightarrow{\theta} i_*i^!A[+1]$$

  Furthermore, the pair made of the functor $i^!$ as well as the 2-triangle above is unique up to an isomorphism.

  2. If $\alpha : A \to B$ is an arrow of $\mathsf{H}(X)$ the morphism

$$i_*i^!(\alpha) : i_*i^!A \to i_*i^!B$$

  is the unique morphism of $\mathsf{H}(Y)$ making the square:

$$\begin{tikzcd} i_*i^!A \arrow[r] \arrow[d, "i_*i^!\alpha"'] & A \arrow[d, "\alpha"] \\ i_*i^!B \arrow[r] & B \end{tikzcd}$$

  commutative. It gies a morphism of distinguished triangles

$$\begin{tikzcd} i_*i^!A \arrow[r] \arrow[d, "i_*i^!\alpha"'] & A \arrow[r] \arrow[d, "\alpha"] & j_*j^!A \arrow[r] \arrow[d, "j_*j^!\alpha"'] & i_*i^!A[+1] \arrow[d, "i_*i^!\alpha"'] \\ i_*i^!B \arrow[r] & B \arrow[r] & j_*j^!B \arrow[r] & i_*i^!B[+1] \end{tikzcd}$$

  3. Finally, the 2-morphism $i^!$ is right adjoint of the 1-morphism i_*. The counit 2-morphism $i_*i^! \to \mathrm{id}$ is the one that is contained in the distinguished 2-triangle. The unit 2-morphism is a 2-isomorphism.
- There exists a 2-functor

$${}^{\mathrm{Imm}}\mathsf{H}^! : (\mathbf{Sch}\,/S)^{\mathrm{Imm}} \to \mathfrak{TR},$$

  uniqute up to an isomorphism, which is a global right adjoint of the 2-functor ${}^{\mathrm{Imm}}H_*$. Denote by $c^!(f,g)$ the 2-isomorphism of connection of this functor.

4.6 The axiom of locality

[Ayoub, $i^!$: $\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M}) \to \mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M})$ Prop.4.5.32, p.543; Th.4.5.34, Rem.4.5.35, p.544; Lem.4.5.43, p.550]

Let $i : Z \to X$ be a closed immersion of S-schemes. Then

- The functor
$$i_* : \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/Z.\mathfrak{M}) \to \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/X.\mathfrak{M})$$
is a left Quillen functor for the injective structures $(\mathbf{W}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{inj})$ (with topology $t_\emptyset$) and $(\mathbf{W}_\tau, \mathbf{Cof}_{inj}, \mathbf{Fib}_{\tau-inj})$ (with topology τ).
- The functor
$$i_* : \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/Z,\mathfrak{M}) \to \mathbf{Shv}_{t_\infty}(\mathrm{Sm}/X,\mathfrak{M})$$
is also a left Quillen functor for the $\mathbb{A}^1$-local injective structures $(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{\mathbb{A}^1-inj})$ (with topology τ). Actually, i_* preserves the $\mathbb{A}^1$-weak equivalences, and so derives trivially:
$$\mathsf{L}i_* \simeq i_* \simeq \mathsf{R}i_*$$

Let X be a S-scheme and $u : U \to X$ a smooth cover for the topology τ. Then

- The functor
$$u^* : \mathbf{PreShv}(\mathrm{Sm}/X,\mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/U,\mathfrak{M})$$
preserves and detects the $\mathbb{A}^1$-weak equivalences.

(And so, whenever a Quillen adjunction involving u^* is available, u^* derives trivially:
$$\mathsf{L}u^* \simeq u^* \simeq \mathsf{R}u^* \text{)}$$

[Ayoub, (Morel-Voevodsky, Th.2.21, p.114), the locality for S^1-spectra, Th. 4.5.36, proof outline, p.545-550]

(This is the locality for S^1-spectra:

$$\begin{aligned}\mathbf{Ho}_{\mathbb{A}^1}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})\right) &\cong \mathbf{Ho}_{\mathbb{A}^1}\left(\mathbf{PreShv}(\mathrm{Sm}/X,\mathfrak{M})\right)\\ &\cong \mathbf{Ho}_{\mathbb{A}^1}\,\mathbf{Spect}_{S^1}\left(\mathbf{PreShv}(\mathrm{Sm}/X)\right) \cong Sh_{S^1}(X)\end{aligned}$$)

If H is a projective cofibrant object of $\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})$. The commutative square

$$\begin{tikzcd} j_\sharp j^* H \arrow[r] \arrow[d] & H \arrow[d] \\ \star \arrow[r] & i_* i^* H \end{tikzcd}$$

is homotopy cocartesian relative to the $\mathbb{A}^1$-local model structure.

Proof outline.

(Lem.4.5.37, p.545) Reduction to the case $H = a_{t_\emptyset}(X' \otimes A_{cst})$: proving the following square ((4.101), p.548):

$$\begin{tikzcd} a_{t_\emptyset}(U' \otimes A_{cst}) \arrow[r] \arrow[d] & a_{t_\emptyset}(X' \otimes A_{cst}) \arrow[d] \\ * \arrow[r] & i_* a_{t_\emptyset}(Z' \otimes A_{cst}) \end{tikzcd}$$

is homotopy cocartesian (using the fact, the functors $j_\sharp, j^*, i^*$ commute with the functor $a_{t_\emptyset}$), where A_{cst} is the constant presheaf valued at a cofibrant object A of $\mathfrak{M}$ and U', Z' are defined by the following commutative diagram of cartesian squares starting with a smooth X-scheme X' :

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d] & X' \arrow[d] & Z' \arrow[l, "i'"'] \arrow[d] \\ U \arrow[r, "j"] & X & Z \arrow[l, "i"'] \end{tikzcd}$$

(Lem.4.5.41, p.548) Reduction to proving the following square is homotopy cocartesian in $\mathbf{Ho}_{\mathbb{A}^1}\left(\mathbf{PreShv}(\mathrm{Sm}/X,\mathfrak{M})\right)$:

$$\begin{tikzcd} U' \otimes A_{cst} \arrow[r] \arrow[d] & X' \otimes A_{cst} \arrow[d] \\ U \otimes A_{cst} \arrow[r] & i_* Z' \otimes A_{cst} \end{tikzcd}$$

i.e. proving the following is an $\mathbb{A}^1$-weak equivalence:

$$(G \otimes A_{cst} \to F \otimes A_{cst}) := \left(\left[X' \coprod_{U'} U\right] \otimes A_{cst} \to i_* Z' \otimes A_{cst}\right)$$

(Cor.4.5.40, p.547) Reduction to proving

$$T_{Y,s} \otimes A_{cst} \to Y \otimes A_{cst}$$

is an $\mathbb{A}^1$-weak equivalence of $\mathbf{PreShv}(\mathrm{Sm}/Y,\mathfrak{M})$, where Y is a X-scheme with a section $s : Y \to F$ and $T_{Y,s} \to Y$ is as below:

$$\text{ttt}\quad T_{Y,s} = p_Y^* G \times_{p_Y^* F} Y \overset{\text{not stated....}}{=} \boxed{s^* G}$$

$$\begin{tikzcd} & T_{Y,s} \arrow[dl] \arrow[dd] \arrow[dr] & & \\ p_Y^* G \arrow[rr] \arrow[dd] & & G = \left[X' \coprod_{U'} U\right] \arrow[dd] \arrow[dr] \arrow[ddr, bend left] & \\ & Y \arrow[dl, bend right, "s"'] \arrow[rr, "p_Y"] \arrow[dr, bend right, "s"] & & X \\ p_Y^* F \arrow[ur] \arrow[rr] & & F \arrow[ur] \arrow[r, equal] & i_* Z' \end{tikzcd}$$

(Just before Prop.4.5.42, p.549) Final reduction to the case $Y = X$.

(Prop.4.5.42, p.549) Completion of the proof Proof of Proposition 4.5.42, which is the final reduction.

□

[Ayoub, Morel-Voevodsky, Th. 4.5.36, p.545]

If H is a projective cofibrant object of $\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M})$. The commutative square

$$\begin{tikzcd} j_\sharp j^* H \arrow[r] \arrow[d] & H \arrow[d] \\ \star \arrow[r] & i_* i^* H \end{tikzcd}$$

is homotopy cocartesian relative to the $\mathbb{A}^1$-local model structure.

[Ayoub, Lem. 4.5.37, p.545]

It suffices to prove Theorem 4.5.36 for

$$H = a_{t_\emptyset}\left(X' \otimes A_{cst}\right)$$

with A a cofibrant object of $\mathfrak{M}$ and X' a smooth X-scheme (and A_{cst} is the constant presheaf valued at A introduced in Proposition 4.4.4).

Rough idea of the proof.

- Projective cofibrations of $\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M})$ are generated by the class C of arrows:

$$a_{t_\emptyset}\left(X' \otimes A_{cst}\right) \to a_{t_\emptyset}\left(X' \otimes B_{cst}\right)$$

  with $A \to B$ a cofibration of $\mathfrak{M}$ whose target is β-accessible (β being a sufficiently large cardinal) and X' a smooth X-scheme.
- Any projective cofibrant H is a retract of $\Phi_{C,\beta}(\emptyset \to H)$ (see Proposition 4.2.26).
- Thus, it suffices to prove Theorem 4.5.36 for

$$\Phi_{C,\beta}(\emptyset \to H) = \mathrm{Colim}_{\nu\in\lambda}\left(\emptyset \to \Psi_1 \to \Psi_2 \to \cdots \to \Psi_\nu \to \Psi_{\nu+1} \to \cdots\right)_{\nu\in\lambda}$$

  which we prove by transfinite induction:

  Case $\mu = \nu + 1$ Then we can prove the claim since $\Psi_\mu = \Psi_{\nu+1}$ is the pushout of the diagram:

$$\begin{tikzcd} \coprod_i a_{t_\emptyset}\left(X'_i \otimes (A_i)_{cst}\right) \arrow[r] \arrow[d] & \Psi_\nu \\ \coprod_i a_{t_\emptyset}\left(X'_i \otimes (B_i)_{cst}\right) & \end{tikzcd}$$

  with $u_i : A_i \to B_i$ the cofibrations of $\mathfrak{M}$ and X'_i the smooth X-schemes.

  Case μ is a limit ordinal Applying Lemma 4.2.69 to

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathsf{Cof}\left(j_\sharp j^*\Psi_1 \to \Psi_1\right) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathsf{Cof}\left(j_\sharp j^*\Psi_\nu \to \Psi_\nu\right) \arrow[r] \arrow[d] & \mathsf{Cof}\left(j_\sharp j^*\Psi_{\nu+1} \to \Psi_{\nu+1}\right) \arrow[r] \arrow[d] & \cdots \\ \emptyset \arrow[r] & i_* i^* \Psi_1 \arrow[r] & \cdots \arrow[r] & i_* i^* \Psi_\nu \arrow[r] & i_* i^* \Psi_{\nu+1} \arrow[r] & \cdots \end{tikzcd}$$

$\square$

[Ayoub, Lem. 4.5.41, p.548]

There exists a commutative square of presheaves of sets:

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d] \\ U \arrow[r] & i_*Z' \end{tikzcd}$$

where the horisontale lower arrow is given by the unique element $\hom(U, i_*Z') = \hom(\emptyset, Z')$. Furthermore, so that the square

$$\begin{tikzcd} a_{t_\emptyset}(U' \otimes A_{cst}) \arrow[r] \arrow[d] & a_{t_\emptyset}(X' \otimes A_{cst}) \arrow[d] \\ * \arrow[r] & i_*a_{t_\emptyset}(Z' \otimes A_{cst}) \end{tikzcd}$$

becomes homotopy cocartesian, it suffices that the following square:

$$\begin{tikzcd} U' \otimes A_{cst} \arrow[r] \arrow[d] & X' \otimes A_{cst} \arrow[d] \\ U \otimes A_{cst} \arrow[r] & i_*Z' \otimes A_{cst} \end{tikzcd}$$

is homotopy cocartesian (in $\mathbf{Ho}_{\mathbb{A}^1}$ $(\mathbf{PreShv}(\mathrm{Sm}\,/X, \mathfrak{M}))$ (of Def.4.5.12)).

Rough idea of the proof.

- The commutativity of the square follows from the fact $\hom(U', i_*Z') = *$.
- The latter claim follows from the fact that the pushout of

$$\begin{tikzcd} U \otimes A_{cst} \arrow[r] \arrow[d] & i_*Z' \otimes A_{cst} \\ * & \end{tikzcd}$$

is canonically identified with $a_{t_\emptyset}(i_*Z' \otimes A_{cst})$ (Lem.4.5.33).

□

[Ayoub, Cor. 4.5.40, p.547]

Let $G \to F$ be a morphism of presheaf of sets on Sm/X. For a section $s : Y \to F$ with Y a smooth X-scheme, we denote by $T_{Y,s}$ the presheaf of sets on Sm/Y defined by

$$T_{Y,s} = p_Y^* G \times_{p_Y^* F} Y \quad \Big(\text{I prefer to thnk as the } \underline{\text{pullback}}\text{: } \boxed{(T_{Y,s} \to Y) = s^*(G \to F)} \Big)$$

with p_Y the structured projection of Y on X and the arrow $Y \to p_Y^* F$ utilised in the fiber product is the adjoint of $(p_Y)_\sharp Y \simeq Y \xrightarrow{s} F$.

tt

$$T_{Y,s} = p_Y^* G \times_{p_Y^* F} Y \overset{\text{not stated....}}{=} \boxed{s^*G}$$

$$\begin{tikzcd}
& T_{Y,s} \arrow[dl] \arrow[dd] \arrow[dr] & & \\
p_Y^* G \arrow[rr] \arrow[dd] & & G \arrow[dd] \arrow[dr] & \\
& Y \arrow[rr, "p_Y"] \arrow[dl, bend right, "s"'] \arrow[dr, bend right, "s"] & & X \\
p_Y^* F \arrow[ur] \arrow[rr] & & F \arrow[ur] &
\end{tikzcd}$$

Let A be a cofibrant object of $\mathfrak{M}$ and suppose that for any smooth X-scheme Y and any section $s \in F(Y)$ the morphism:

$$T_{Y,s} \otimes A_{cst} \to Y \otimes A_{cst}$$

is an $\mathbb{A}^1$-weak equivalence of $\mathbf{PreShv}(\mathrm{Sm}/Y, \mathfrak{M})$. Then

$$G \otimes A_{cst} \to F \otimes A_{cst}$$

is also an $\mathbb{A}^1$-weak equivalence.

Rough idea of the proof.

- (By Theorem 4.4.61, Th.4.5.11) The continuous functor $p_F : (\mathrm{Sm}/X)/F \to \mathrm{Sm}/X$ induces a Quillen adjunction:
$$((p_F)_\sharp, (p_F)^*) : \mathbf{PreShv}\left((\mathrm{Sm}/X)/F, \mathfrak{M}\right) \to \mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})$$
for the $\mathbb{A}^1$-local projective structures.projective structures.)
- (By Corollary 4.5.39) The arrow
$$G \otimes A_{cst} \to F \otimes A_{cst}$$
can be identified with the result of the functor $\mathsf{L}\,(p_F)_\sharp$ applied to the arrow:
$$(G_F \otimes A_{cst} \to * \otimes A_{cst}) \in \mathbf{PreShv}\left((\mathrm{Sm}/X)/F, \mathfrak{M}\right)$$
- Observe that the $\mathbb{A}^1$-weak equialences in $\mathbf{PreShv}\left((\mathrm{Sm}/X)/F, \mathfrak{M}\right)$ are detected by:

$$\prod_{(Y/X)\to F \in \mathrm{Ob}((\mathrm{Sm}/X)/F)} ((Y/X) \to F)^* : \mathbf{PreShv}\left((\mathrm{Sm}/X)/F, \mathfrak{M}\right) \to \prod_{(Y/X)\to F \in \mathrm{Ob}((\mathrm{Sm}/X)/F)} \mathbf{PreShv}(\mathrm{Sm}/Y, \mathfrak{M})$$

- Thus, to show
$$G \otimes A_{cst} \to F \otimes A_{cst}$$
is a $\mathbb{A}^1$-weak equialences, it suffices to show
$$s^*\left(G_F \otimes A_{cst} \to * \otimes A_{cst}\right) \in \mathbf{PreShv}\left((\mathrm{Sm}/Y, \mathfrak{M}\right)$$
is an $\mathbb{A}^1$-equivalence, for any section $(s : (Y/X) \to F) \in \mathrm{Ob}\left((\mathrm{Sm}/X)/F\right)$.
- Finally, the proof is complete by observing
$$s^*\left(G_F \otimes A_{cst} \to * \otimes A_{cst}\right)$$
is given by
$$T_{Y,s} \otimes A_{cst} \to Y \otimes A_{cst}$$

□

[Ayoub, Just before Prop.4.5.42, p.549]

Recall the commutative diagram of cartesian squares starting with a smooth X-scheme X' :

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d] & X' \arrow[d, "\text{smooth}"] & Z' \arrow[l, "i'"'] \arrow[d] \\ U \arrow[r, "j", "\text{complementary open}"'] & X & Z \arrow[l, "i"', "\text{closed embedding}"] \end{tikzcd}$$

- For $W \in \mathrm{Sm}/X$,

$$\left[X' \coprod_{U'} U\right](W) = \hom_X(W, X') \coprod_{\hom_X(W,U')} \hom_X(W,U) = \begin{cases} \hom_X(W,X') & \text{if } W \times_X Z \neq \emptyset \\ * & \text{if } W \times_X Z = \emptyset \end{cases}$$

Proof.

The case $W \times_X Z \neq \emptyset$ $\hom_X(W,U) = \hom_X(W,U') = \emptyset$, and so the coproduct is clearly $\hom_X(W,X')$.

The case $W \times_X Z = \emptyset$ $\hom_X(W,U') = \hom_X(W,X')$ and $\hom_X(W,U) = *$, and so the coproduct is clearly $*$.

□

- Now, for each $s : Y \to i_*Z$, consider the pullback construction $\boxed{(T_{Y,s} \to Y) = s^*(G \to F)}$ for $(G \to F) = (X' \coprod_{U'} U \to i_*Z')$, for which, Ayoub used more detaild symbol $T_{X',Y,s}$ to denote $T_{Y,s}$. Then for any $P \in \mathrm{Sm}/Y$,

$$T_{X',Y,s}(P \to Y) = \begin{cases} \hom_X(P,X') \times_{\hom_Z(P\times_X Z, Z')} * & \text{if } P \times_X Z \neq \emptyset \\ * & \text{if } P \times_X Z = \emptyset \end{cases}$$

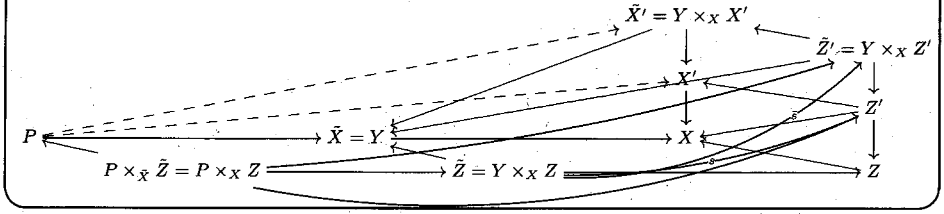
Proof. From the pullback characterization of $T_{X',Y,s}$: $\begin{tikzcd} T_{X',Y,s} \arrow[r] \arrow[d] & X' \coprod_{U'} U \arrow[d] \\ Y \arrow[r, "s"] & i_*Z' \end{tikzcd}$ and the above calculation of $\left[X' \coprod_{U'} U\right](W)$, we have the following pullback of sets:

$$\begin{tikzcd} T_{X',Y,s}(P \to Y) \arrow[r] \arrow[d] & \boxed{\begin{cases} \hom_X(P,X') & \text{if } P \times_X Z \neq \emptyset \\ * & \text{if } P \times_X Z = \emptyset \end{cases}} \arrow[d] \\ * = \{p_P : P \to Y\} \arrow[r] & \left(\begin{aligned} \hom_X(P, i_*Z') &= \hom_Z(i^*P, Z') \\ &= \hom_Z(P \times_X Z, Z') \end{aligned}\right) \end{tikzcd}$$

From this, the claim follows.

□

- Lift everything by $p_Y : Y \to X$ and denote by $\widetilde{(-)}$ the object obtained by lifting $(-)$. Then construct the following commutative diagram, in which the solution set of the dotted arrows $P -- > X'$ makeing relevant faces commute is nothing but $T_{X';Y,s}(P \to Y)$. However, contemplating on this diagram, we find this sokution set is the as the solution set of those dotted arrows $P -- > \tilde{X}' = Y \times_X X'$, which is nothing but $T_{\tilde{X}';\tilde{X},\tilde{s}}(P \to Y = \tilde{X})$, where $\tilde{s} : \tilde{X} = Y \to \tilde{i}_*\tilde{Z}' \left(\to \tilde{i}_*\tilde{X}'\right)$ is the adjoint of $\tilde{s} : \tilde{i}^*\tilde{X} = \tilde{Z} = Y \times_X Z \xrightarrow{1_Y \times s} Y \times_X Z' = \tilde{Z}' \left(\to \tilde{X}'\right)$. Thus, it suffices to treat this case, which is, after rewriting those $\widetilde{(-)}$ by simply $(-)$, treated in Proposition 4.5.42.



[Ayoub, Prop. 4.5.42, p.549]

Let X' be a smooth X-scheme and $s : Z \to X'$ a partial section defined on Z. Denote by $T_{X',s}$ the presheaf of sets (on Sm/X) defined by:

$$T_{X',s}(P) = \begin{cases} \hom_X(P, X') \times_{\hom(P \times_X Z, Z')} * & \text{if } P \times_X Z \neq \emptyset \\ * & \text{if } P \times_X Z = \emptyset \end{cases}$$

Then the arrow

$$T_{X',s} \otimes A_{cst} \to X \otimes A_{cst}$$

is an $\mathbb{A}^1$-weak equivalence.

Rough idea of the proof.

Step 1 Reduction to the case:

- X is affine,
- $s : Z \to X'$ admits an affine neighborhood in X',
- The normal sheaf $\mathcal{N}_s$ of the immersion s is free.

To see this, we consider a family of étale morphisms $(u_i : X_i \to X)_i$, which is a covering for the topology τ. For each i, as in 4.5.1, p.532; proof of Proposition 4.5.4, p.533, $u_i : X_i \to X$ induces

$$\begin{aligned} u_i : \mathrm{Sm}/X &\to \mathrm{Sm}/X_i \\ (P \to X) &\mapsto (P \times_X X_i \to X_i), \end{aligned}$$

which admits a right adjoint

$$\begin{aligned} c_{u_i} : \mathrm{Sm}/X_i &\to \mathrm{Sm}/X \\ (P_i \to X_i) &\mapsto (P_i \to X_i \to X) \end{aligned}$$

with the identification:

$$u_i^* = (c_{u_i})_*.$$

Now, Lemma 4.5.43, p.550, implies we can reduce our proof to showing

$$u_i^* \left(T_{X',s} \otimes A_{cst} \to X \otimes A_{cst}\right)$$

is an $\mathbb{A}^1$-weak equivalence.

However, for a smooth X_i-scheme P_i, we have:

$$\begin{aligned} (u_i^*(T_{X',s})(P_i \to X_i) &= \left((c_{u_i})_* (T_{X',s})\right)(P_i \to X_i) \\ &= T_{X',s}\left((c_{u_i})(P_i \to X_i)\right) = T_{X',s}(P_i \to X_i \to X), \end{aligned}$$

which, by contemplating the commutative diagram below, we can identify with:

$$T_{X'_i,s_i}(P_i \to X_i).$$

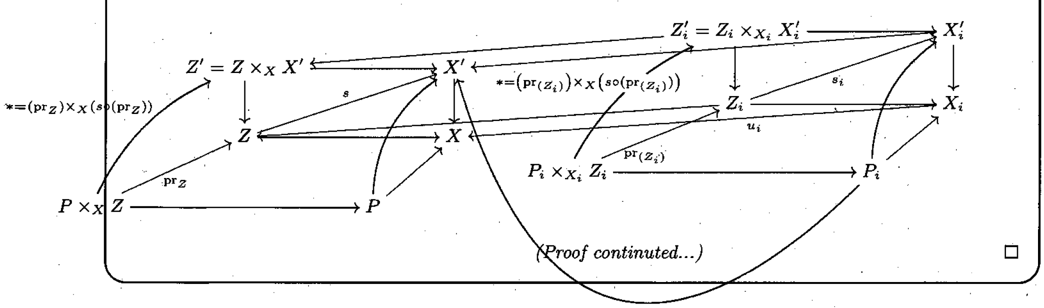
Therefore,

$$u_i^*(T_{X',s}) = T_{X'_i,s_i}$$

and so

$$u_i^*\left(T_{X',s} \otimes A_{cst} \to X \otimes A_{cst}\right) = \left(T_{X'_i,s_i} \otimes A_{cst} \to X_i \otimes A_{cst}\right)$$

Thus, it suffices to prove the claim for the case stated.



(*Proof continued...*) □

[Ayoub, Prop. 4.5.42, p.549]

Let X' be a smooth X-scheme and $s : Z \to X'$ a partial section defined on Z. Denote by $T_{X',s}$ the presheaf of sets (on Sm/X) defined by:

$$T_{X'.s}(P) = \begin{cases} \hom_X(P, X') \times_{\hom(P\times_X Z, Z')} * & \text{if } P \times_X Z \neq \emptyset \\ * & \text{if } P \times_X Z = \emptyset \end{cases}$$

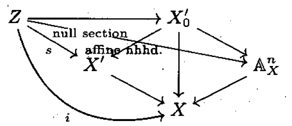
Then the arrow
$$T_{X',s} \otimes A_{cst} \to X \otimes A_{cst}$$
is an $\mathbb{A}^1$-weak equivalence.

Rough idea of the proof (continuted).

Step 2 Reduction to the case:

- X is affine,
- $X' = \mathbb{A}^n_X$,
- $s : Z \to X' = \mathbb{A}^n_X$ is the null section,

To see this, starting with the reduction of Step 1, let us construct



Here X'_0 is an affine neighborhood of s in X' and $Z \to \mathbb{A}^n_X$ is the null section. We would like to reduce to the stated case from that of Step 1 in two steps:

$$\begin{tikzcd} Z \arrow[r, "s"] \arrow[dr, "i"'] & X' \arrow[d] \\ & X \end{tikzcd} \implies \begin{tikzcd} Z \arrow[r] \arrow[dr, "i"'] & X'_0 \arrow[d] \\ & X \end{tikzcd} \implies \begin{tikzcd} Z \arrow[r, "\text{null section}"] \arrow[dr, "i"'] & \mathbb{A}^n_X \arrow[d] \\ & X \end{tikzcd}$$

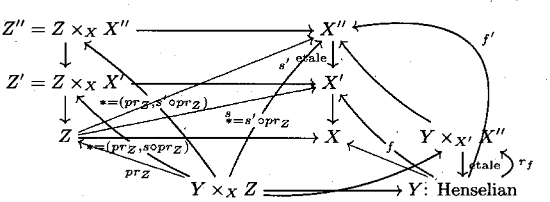
For this, it suffices to prove that for any commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "s'"] \arrow[dr, "s"'] & X'' \arrow[d, "r:\text{etale}"] \\ & X' \end{tikzcd}$$

with $r : X'' \to X'$ etale, the evident morphism of presheaf
$$T_{X'',s'} \to T_{X',s}$$
(whose construction can be seen using the diagram below) becomes an isomorphism upon the sheafication with respect to the topology τ. For this, we may suppose τ is Nisnevich and it suffices to construct an inverse map:
$$T_{X',s}(Y) \to T_{X'',s'}(Y)$$
for each Henselian localisation $Y \to X$ of a point in a smooth X-scheme with $Y \times_X Z \neq \emptyset$.
Then, given $f \in T_{X',s}(Y)$, so $f : Y \to X$, we may construct the uniqute $f' \in T_{X'',s'}(Y)$ as $f' : Y \to X''$ by constructing the commutative diagram below. Here, the condition:

$$Y \times_{X'} X'' \xrightarrow{\text{etale}} Y\text{: etale}$$

was used to uniqutely lift the map $Y \times_X Z \to Y \times_{X'} X''$ to $r_f : Y \to Y \times_{X'} X''$:



(Proof continuted...) □

[Ayoub, Prop. 4.5.42, p:549]

Let X' be a smooth X-scheme and $s : Z \to X'$ a partial section defined on Z. Denote by $T_{X',s}$ the presheaf of sets (on Sm/X) defined by:

$$T_{X',s}(P) = \begin{cases} \hom_X(P, X') \times_{\hom(P \times_X Z, Z')} * & \text{if } P \times_X Z \neq \emptyset \\ * & \text{if } P \times_X Z = \emptyset \end{cases}$$

Then the arrow

$$T_{X',s} \otimes A_{cst} \to X \otimes A_{cst}$$

is an $\mathbb{A}^1$-weak equivalence.

Rough idea of the proof (continuted).

Step 3 Completion of the proof: Proof that

$$T_{\mathbb{A}^n_X,0} \otimes A_{cst} \to X \otimes A_{cst},$$

which is the case reduced in Step 2, is an $\mathbb{A}^1$-weak equivalence.

To see this, it suffices to construct a homotopy between the identity of $T_{\mathbb{A}^n_X,0}(Y)$ and the null map, which is provided by:

$$\begin{aligned} T_{\mathbb{A}^n_X,0}(Y) \times \mathbb{A}^1_X(Y) &\to T_{\mathbb{A}^n_X,0}(Y) \\ (f,t) &\mapsto \left(Y \xrightarrow{(f,t)} \mathbb{A}^n_X \times \mathbb{A}^1_X \xrightarrow{(x_1,\ldots,x_n,t') \mapsto (t'x_1,\ldots,t'x_n)} \mathbb{A}^n_X \right) \end{aligned}$$

(Proof completed!) □

[Ayoub, Lem. 4.5.38, p.546; Cor. 4.5.39, p.547]

- Let F be a presheaf of sets on a small category $\mathcal{S}$. We define

$$\mathrm{N}(F) \in \mathrm{Ob}\left(\Delta^{op}\left(\mathbf{PreShv}(\mathcal{S})\right)\right)$$

which:

- to $\underline{\mathbf{n}} \in \mathrm{Ob}(\Delta)$ associate:

$$\coprod_{U=(U(0)\to\cdots\to U(n)):\underline{\mathbf{n}}\to\mathcal{S}/F} U(0)$$

- to an increading function $r : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ associate the coproduct ofarrows:

$$U(0) \to U(r(0)) \to \coprod_{V:\underline{\mathbf{m}}\to\mathcal{S}/F} V(0)$$

where the second arrow is identified with the corresponding component of the composite functor: $\underline{\mathbf{m}} \to \underline{\mathbf{n}} \to \mathcal{S}/F$.

We will see $\mathrm{N}(F)$ as a presheaf of sets on $\Delta \times \mathcal{S}$.
Denote by $p : \Delta \times \mathcal{S} \to \mathcal{S}$ the projection on the second factor. Then for any cofibrant object A of $\mathfrak{M}$, the evident morphism:

$$\mathsf{L}p^*\left(\mathrm{N}(F) \otimes A_{cst}\right) \to F \otimes A_{cst}$$

is an isomorphism in $\mathbf{Ho}\left(\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})\right)$.

- Let $\mathcal{S}$ be a small category and $G \to F$ a morphism of presheaves of sets on $\mathcal{S}$. Denote by G_F the presheaf of sets on $\mathcal{S}/F$ wwhich to an arrow $s : U \to F \in \mathrm{Ob}(\mathcal{S}/F)$ associates the fiber product $G(U) \times_{F(U)} *$ with $* \to F(U)$ the application which points the section s.
Denote by $p_F : \mathcal{S}/F \to \mathcal{S}$ the evident functor and

$$p_F^* : \mathbf{PreShv}(\mathcal{S}, -) \to \mathbf{PreShv}\left(\mathcal{S}/F, -\right)$$

the functor of the right composition by p_F.

(This is the convention of the derivateur of Grothendieck. In the usual notation, this is of course written as follows:

$$(p_F)_* : \mathbf{PreShv}(\mathcal{S}, -) \to \mathbf{PreShv}\left(\mathcal{S}/F, -\right).$$

Here, in the proof of Corollary 4.5.38, and possibly in some other places, the notations of Grothendieck'derivateurs are employed:

Convention	The right composition	Its left adjoint
The usual	p_*	p^*
Grothendieck's derivateur	p^*	$p_\sharp$

)

For any cofibrant object A of $\mathfrak{M}$, the evident morphism

$$\mathsf{L}(p_F)_\sharp\left(G_F \otimes A_{cst}\right) \to G \otimes A_{cst}$$

This is in Grothendieck's derivateur notation. In the usal notation, this is written as follows:

$$\mathsf{L}(p_F)^*\left(G_F \otimes A_{cst}\right) \to G \otimes A_{cst}$$

is invertaible in $\mathbf{Ho}\left(\mathbf{PreShv}(\mathcal{S}, \mathfrak{M})\right)$.

[Ayoub, Lem. 4.5.43, p.550]

Let X be a S-scheme and $u : U \to X$ a smooth cover for the topology τ. The functor

$$u^* : \mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/U, \mathfrak{M})$$

preserves and detects the $\mathbb{A}^1$-weak equivalences.

Forgotten (?) to be stated by Ayoub, because these are commen seneses?

Consider a smooth morphism $U \to X$ which is open for the topology τ (e.g. the complementary open immersion of a closed imersion).

- By Lemma 4.5.43 and its proof, which used Def.4.4.60, Th.4.4.61; Prop.4.5.4 and its proof; and The.4.5.10, we have a Quillen adjunction:

$$(j_\sharp, j^*) = ((c_j)^*, (c_j)_*) : \mathbf{PreShv}(\mathrm{Sm}/U, \mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})$$

  for the structure $\left(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\mathbb{A}^1}\right)$.
- $j^* = (c_j)_* : \mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M}) \to \mathbf{PreShv}(\mathrm{Sm}/U, \mathfrak{M})$ preserves $\mathbf{W}_{\mathbb{A}^1}$, and so j^* drives trivially.
- Thus, the above Quillen adjunction induces the adjunction:

$$(\mathsf{L}j_\sharp, j^*) : \mathbf{Ho}\left(\mathbf{PreShv}(\mathrm{Sm}/U, \mathfrak{M})\right) \to \mathbf{Ho}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)$$

- Furthermore, levelwise application of above $(j_\sharp, j^*) = ((c_j)^*, (c_j)_*)$ induces a Quillen adjunction:

$$(j_\sharp, j^*) = ((c_j)^*, (c_j)_*) : \mathbf{Spect}^{\Sigma}_{T_U}\left(\mathbf{PreShv}(\mathrm{Sm}/U, \mathfrak{M})\right) \to \mathbf{Spect}^{\Sigma}_{T_X}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)$$

  for the structure $\left(\mathbf{W}_{\mathbb{A}^1-st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\mathbb{A}^1-st}\right)$.

Actually, these are special cases of Th.4.5.14 and its stable generalisation Th.4.5.23 obtained using Lem.4.3.34. Although we have to show $j^*\omega^p_H$ is an $\mathbb{A}^1$-weak equivalence, this is easier than that of i_* for a closed immersion i treated in Th.4.5.45 and its proof. Actually, in the analogous commutative diagram:

$$\begin{tikzcd}
j^*\,\mathrm{Sus}^{p+1}_{T_X,\Sigma}(T_X \otimes H) \arrow[r, "j^*\omega^p_H"] \arrow[d] & j^*\,\mathrm{Sus}^{p}_{T_X,\Sigma}(H) \arrow[d] \\
\mathrm{Sus}^{p+1}_{T_U,\Sigma}(T_U \otimes j^*H) \arrow[r, "\omega^p_{j^*H}"] & \mathrm{Sus}^{p+1}_{T_U,\Sigma}(j^*H)
\end{tikzcd},$$

where

$$T_X = p_X^*T,\ T_U = p_U^*T = (U \to X)^*p_X^*T = (U \to X)^*T_X$$

via the commutative diagram $\begin{tikzcd}[column sep=small] U \arrow[rr] \arrow[dr, "p_U"'] & & X \arrow[dl, "p_X"] \\ & X & \end{tikzcd}$, we can easily deduce the $\mathbb{A}^1$-weak equivalence property of $j^*\omega^p_H$ from that of $\omega^p_{j^*H}$. This is because the vertical morphisms are isomorphisms, which are essentially canonical isomorphisms:

$$j^*\left(T_X^{\otimes r} \otimes H\right) \xrightarrow{\sim} T_U^{\otimes r} \otimes j^*H$$

Together with the easier part of Lemma 4.5.43, this argument indicates j^* derives trivially even for the stable case with the structure $\left(\mathbf{W}_{\mathbb{A}^1-st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\mathbb{A}^1-st}\right)$.

- Thus, we obtain an adjunction:

$$(\mathsf{L}j_\sharp, j^*) : \mathbf{Ho}\left(\mathbf{Spect}^{\Sigma}_{T_U}\left(\mathbf{PreShv}(\mathrm{Sm}/U, \mathfrak{M})\right)\right) \to \mathbf{Ho}\left(\mathbf{Spect}^{\Sigma}_{T_X}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)\right)$$

  induced by the structure $\left(\mathbf{W}_{\mathbb{A}^1-st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\mathbb{A}^1-st}\right)$.
- Also, by Theorem 4.5.23, we obtain the adjunction:

$$(j^*, \mathsf{R}j_*) : \mathbf{Ho}\left(\mathbf{Spect}^{\Sigma}_{T_X}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)\right) \to \mathbf{Ho}\left(\mathbf{Spect}^{\Sigma}_{T_U}\left(\mathbf{PreShv}(\mathrm{Sm}/U, \mathfrak{M})\right)\right)$$

  induced by the structure $\left(\mathbf{W}_{\mathbb{A}^1-st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\mathbb{A}^1-st}\right)$.
- For a cofibrant object $\mathbf{E}$ in $\mathbf{Spect}^{\Sigma}_{T_X}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)$, we may take $j_\sharp j^*\mathbf{E}$ for $\mathsf{L}j_\sharp j^*\mathbf{E} \in \mathbf{Ho}\left(\mathbf{Spect}^{\Sigma}_{T_X}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)\right)$. This is because $j^*\mathbf{E}$ is still cofibrant, for (j^*, j_*) is a Quillen adjunction, and so we do not have to derive $j_\sharp$ for this element.

[Ayoub, Cor. 4.5.44 and its proof, p.550]

Let $i : Z \to X$ be a closed immersion of S-schemes and j the complementary open immersion.

- Then the couple of functors $(\mathsf{L}i^*, j^*)$ is conservative on $\mathbf{Ho}_{\mathbb{A}^1}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)$.
- Furhtermore, the counit morphism $\mathsf{L}i^*\mathsf{R}i_*$ is invertible. (We should derive i_* and so, we should use $\mathsf{R}i_*$ here, because, unlike Prop.4.5.34, Rem.4.5.35, we are not restricting to t-sheaves.)

[Ayoub, $i^!: \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})) \to \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_Z)}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M}))$ Prop.4.5.45, p.551; Lem.4.5.46, p.552]

Let $i: Z \to X$ be a closed immersion of S-schemes. Then the functor

$$i_*: \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_Z)}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M})) \to \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})),$$

obtained by applying:

$$\begin{cases} i_* \text{ levelwise;} \\ \text{the natural transformation } a_{t_\emptyset}(T_X) \otimes i_*(-) \to i_*(a_{t_\emptyset}(T_Z) \otimes -)\text{, to the structure maps,} \end{cases}$$

is a left Quillen functor with respect to the stable injective structures on the spectra induced from the $\mathbb{A}^1$-local injecture structures $(\mathbf{W}_{\mathbb{A}^1}, \mathbf{Cof}_{inj}, \mathbf{Fib}_{inj-\mathbb{A}^1})$ on the categories of t-sheaves. (Unlike the left Quillen functor j^* for the complementary open immersion treated after Prop.4.5.43, this is NOT covered by Th.4.5.23).

Rough outline of the proof.

- By Def.4.1.24, Prof.4.1.23; Def.4.3.29; Def.4.5.12, it suffices to:
  - construct a right adjoint of i_*;
  - show i_* preserves the stable injective cofibrations and stable injective $\mathbb{A}^1$-trivial cofibrations.
- We can construct the right adjoint $i^!$ of i_* : For
$$\mathbf{E} \in \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X,\mathfrak{M})),$$
define $i^!\mathbf{E}$ by:
$$\begin{cases} i^!\mathbf{E} & := \mathbf{F} = (\mathbf{F}_n)_{n\in\mathbb{N}} \in \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_Z)}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M})), \\ \mathbf{F}_n & := \mathrm{Eq}\left(\prod_{k\in\mathbb{N}} \underline{\mathrm{Hom}}_g(a_{t_\emptyset}(T_Z)^{\otimes k}, i^!\mathbf{E}_{k+n}) \rightrightarrows \prod_{l\in\mathbb{N}} \underline{\mathrm{Hom}}_g(a_{t_\emptyset}(T_Z)^{\otimes l}, i^!\underline{\mathrm{Hom}}_g(a_{t_\emptyset}(T_X), \mathbf{E}_{l+1+n}))\right) \end{cases}$$
where Σ_n operates on $\mathbf{F}_n$ by the following restriction $1 \times \Sigma_n \subset \Sigma_m \times \Sigma_n \subset \Sigma_{m+n}$ of the action of Σ_{m+n} on $\mathbf{E}_{m+n}$. The two arrows in this equalizer are given by the adjoints of the assemby morphism of $a_{t_\emptyset}(T_Z)$-spectrum $\mathbf{E}$ and by the composite:
$$\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k}, i^!(-)\right) \simeq \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k-1}, \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z), i^!(-)\right)\right) \to \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k-1}, i^!\underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_X), -\right)\right)$$
We can easily verify that the product of arrows:
$$\mathrm{F}_n \to \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k}, i^!\mathbf{E}_{k+n}\right) \simeq \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z), \underline{\mathrm{Hom}}_g\left(a_{t_\emptyset}(T_Z)^{\otimes k-1}, i^!\mathbf{E}_{k-1+1+n}\right)\right)$$
factors through the sub-object $\underline{\mathrm{Hom}}_g(a_{t_\emptyset}(T_Z), \mathbf{F}_{n+1})$.
We thus obtain a symmetric $a_{t_\emptyset}(T_Z)$-spectrum $\mathbf{F}$, for which we can verify the funcor $\hom(i_*(-), \mathbf{E})$ is represented by $\mathbf{F}$.
- By Theorem 4.5.34, i_* levelwise preserves injective cofibrations and injective $\mathbb{A}^1$-trivial cofibrations.
- To strengthen the above preservation property to the stable setting as in Definition 4.3.29, we shoudl verify that, for any injective cofibrant $K \in \mathrm{Ob}(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z,\mathfrak{M}))$ and $p \in \mathbb{N}$, i_* sends ω^p_K of page 484 to stable $\mathbb{A}^1$-equivalences. To see this, consider:
$$\begin{array}{ccc} \mathrm{Sus}^{p+1}_{a_{t_\emptyset}(T_X),\Sigma}(a_{t_\emptyset}(T_X) \otimes i_*K) & \xrightarrow{\omega^p_{i_*K}} & \mathrm{Sus}^p_{a_{t_\emptyset}(T_X),\Sigma}(i_*K) \\ \downarrow & & \downarrow \\ i_*\,\mathrm{Sus}^{p+1}_{a_{t_\emptyset}(T_Z),\Sigma}(a_{t_\emptyset}(T_Z) \otimes K) & \xrightarrow{i_*\omega^p_K} & i_*\,\mathrm{Sus}^p_{a_{t_\emptyset}(T_Z),\Sigma}(K) \end{array}$$
To see $i_*\omega^p_K$ is an $\mathbb{A}^1$-equivalence, for $\omega^p_{i_*K}$ is so, it suffices to show the vertical arrows are levelwise $\mathbb{A}^1$-weak equivalences. Thus, suffices to show the arrows:
$$a_{t_\emptyset}(T_X^{\otimes r}) \otimes i_*K \to i_*(a_{t_\emptyset}(T_Z^{\otimes r}) \otimes K)$$
are $\mathbb{A}^1$-weak equivalences.
- To see this, let j be the immersion of the complementary open, and we apply the "unstable" conservation theorem w.r.t. $(\mathrm{L}i^*, j^*)$ of Corollary 4.5.44:

$\underline{j^*}$ $j^*\left[a_{t_\emptyset}(T_X^{\otimes r}) \otimes i_*K\right] \to j^*\left[i_*\left(a_{t_\emptyset}(T_Z^{\otimes r}) \otimes K\right)\right]$ is invertible.

Proof. This is the unique arrow between null objects for $j^*i_* = 0$. □

$\underline{\mathrm{L}i^*}$ $\mathrm{L}i^*\left[a_{t_\emptyset}(T_X^{\otimes r}) \otimes i_*K\right] \to \mathrm{L}i^*\left[i_*\left(a_{t_\emptyset}(T_Z^{\otimes r}) \otimes K\right)\right]$ is invertible.

Proof. For this, consider the commutative diagram:
$$\begin{array}{ccc} \mathrm{L}i^*\left[a_{t_\emptyset}(T_X^{\otimes r}) \otimes i_*K\right] & \longrightarrow & \mathrm{L}i^*\left[i_*\left(a_{t_\emptyset}(T_Z^{\otimes r}) \otimes K\right)\right] \\ \wr\downarrow & & \downarrow \\ a_{t_\emptyset}(T_Z^{\otimes r}) \otimes \mathrm{L}i^*i_*K & \longrightarrow & a_{t_\emptyset}(T_Z^{\otimes r}) \otimes K \end{array}$$
Then the claim follows from the latter part of Cor.4.5.44 which claims $\mathrm{L}i^*i_*$ is invertible. (Here, we need not to derive i_* for we are considering the sheaves like Prop.4.5.34, Rem.4.5.35, unlike Cor.4.5.44).

[Ayoub, Locality for the stable case, Cor. 4.5.47, p.552]

Let $i : Z \to X$ be a closed immersion of S-schemes and j the complementary open immersion. Then (we obtain an distinguished 2-triangle in $\mathbf{SH}^T_{\mathfrak{M}}(X)$:

$$\mathsf{L}j_\sharp j^* \to \mathrm{id} \to \mathsf{R}i_*\mathsf{L}i^* \to \cdots ,$$

and so) the couple of funcotrs $(\mathsf{L}i^*, j^*)$ is conservative on $\mathbf{SH}^T_{\mathfrak{M}}(X)$.
Furthermore, the counit morphism $\mathsf{L}i^*\mathsf{R}i_*$ is invertible.

Rough outline of the proof.

- For a projectively cofibrant $a_{t_\emptyset}(T_X)$-spectrum valueed in $\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M})$, the commutative square :

$$\begin{tikzcd} j_\sharp j^*\mathbf{E} \arrow[r] \arrow[d] & \mathbf{E} \arrow[d] \\ * \arrow[r] & i_*i^*\mathbf{E} \end{tikzcd}$$

  is a levelwise homotopy cocartesian by Proposition 4.5.36. Thus, it is a homotopy cocartesian of spectra.

  A proof of this fact:

  If a commutative square of spectra is levelwise a homotopy cocartesian square of spaces, it is homotopy cocartesian.

  may be found for instance in the proof of Lemma 2.6 of

  [LRV] Wolfgang Lueck, Holger Reich, Marco Varisco, Commuting homotopy limits and smash products, K-Theory 30-2 (2003) 137-165.

- By Proposition 4.5.45,

$$i_* : \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_Z)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/Z, \mathfrak{M})\right) \to \mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M})\right)$$

  preserves the stable $\mathbb{A}^1$-equivalences, and so derives trivially.

- Then, for a cofibrant object $\mathbf{E}$ in $\mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M})\right)$, we may take

$$i_*i^*\mathbf{E}$$

  for $\mathsf{R}i_*\mathsf{L}i^*\mathbf{E} \in \mathbf{Ho}\left(\mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M})\right)\right)$. This is because:

  - we do not have to derive i^* for this cofibrant object $\mathbf{E}$, for we have a Quillen adjunction (i^*, i_*);
  - we do not have to derive i_* we so above.

- As we discussed between Lem.4.5.43 and Cor.4.5.44, for a cofibrant object $\mathbf{E}$ in $\mathbf{Spect}^{\Sigma}_{a_{t_\emptyset}(T_X)}\left(\mathbf{Shv}_{t_\emptyset}(\mathrm{Sm}/X, \mathfrak{M})\right)$, and so also a cofibrant object in $\mathbf{Spect}^{\Sigma}_{T_X}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)$, we may take

$$j_\sharp j^*\mathbf{E}$$

  for $\mathsf{L}j_\sharp j^*\mathbf{E} \in \mathbf{Ho}\left(\mathbf{Spect}^{\Sigma}_{T_X}\left(\mathbf{PreShv}(\mathrm{Sm}/X, \mathfrak{M})\right)\right)$. This is because $j^*\mathbf{E}$ is still cofibrant, for (j^*, j_*) is a Quillen adjunction, and so we do not have to derive $j_\sharp$ for this object $j^*\mathbf{E}$.

- By the above discussion, we obtain an distinguished 2-triangle in $\mathbf{SH}^T_{\mathfrak{M}}(X)$:

$$\mathsf{L}j_\sharp j^* \to \mathrm{id} \to \mathsf{R}i_*\mathsf{L}i^* \to \cdots ,$$

  and so, $(\mathsf{L}i^*, j^*)$ is conservative on $\mathbf{SH}^T_{\mathfrak{M}}(X)$.

- The invertibility of the counit morphism $\mathsf{L}i^*\mathsf{R}i_*$ follows from the analogous unstable result of Corollary 4.5.44 by Lem.4.3.59, becase i_* derives trivially for the t-sheaves.

□

4.7 Finishing the verification of the stable homotopy 2-functor axiom DerAlg 5 of $\mathrm{Dia\,Sch}/S \ni (\mathcal{F},\mathcal{I}) \to \mathbb{SH}^{T}_{\mathcal{M}}(\mathcal{F},\mathcal{I}) \in \mathfrak{TR}$ - smooth base change and homotopy invariance

smoth base change [Ayoub, Th.4.5.48, p.553]

Given a cartesian square of S-schemes:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f smooth. The base change morphism $\mathbf{L}f'_\sharp \mathbf{L}g'^* \to \mathbf{L}g^* \mathbf{L}f_\sharp$ is invertible.

Rough outline of the proof.

- Given four functor $f_\sharp, g^*, g'^*$ and $f'_\sharp$ are left Quillen w.r.t. the $\mathbb{A}^1$-stable projective structure $(\mathbf{W}_{\mathbb{A}^1-st}, \mathbf{Cof}_{proj}, \mathbf{Fib}_{proj-\mathbb{A}^1-st})$, it suffices to prove that the natural trasformation $f'_\sharp g'^* \to g^* f_\sharp$ is levelwise w.r.t. the $\mathfrak{M}$ valued presheaves.
- Since the four functors in question commute with small colimits, it suffices to evaluate $U' \otimes A_{cst}$ with U' a smooth X'-scheme and $A \in \mathrm{Ob}(\mathfrak{M})$.
- We then obtain the arrow:

$$\left(U' \otimes_{X'} Y' \to Y' \to Y\right) \otimes A_{cst} \to \left((U' \to X' \to X) \times_X Y\right) \otimes A_{cst}$$

Now the result follows from the transitivity of fiber product of schemes.

□

homotopy invariance [Ayoub, Prop.4.5.49, p.553]

Let X be a S-scheme and denote $p : \mathbb{A}^1_X \to X$ the projection of the affine line on $\mathbb{A}_X$. Then the unit morphism $\mathrm{id} \to \mathbf{R}p_* \mathbf{L}p^*$ is invertible.

Rough outline of the proof.

- It is equivalent to prove the counit $\mathbf{L}p_\sharp \mathbf{L}p^* \to \mathrm{id}$ is invertible.
- By the projection formula, we deduce an isomorphism $\mathbf{L}p_\sharp \mathbf{L}p^*(-) \simeq \left(\mathbf{L}p_\sharp \mathbf{1}_{\mathbb{A}^1_X}\right) \otimes (-)$ with $\mathbf{1}_{\mathbb{A}^1_X}$ the unit object of the monoidal category $\mathbb{M}_T\left(\mathbb{A}^1_X\right)$.
- By definition of $p_\sharp$, the object $p_\sharp \mathbf{1}_{\mathbb{A}^1_X}$ is the suspension T_X-spectrum $\mathrm{Sus}^0_{T_X,\Sigma}\left(\mathbb{A}^1_X \otimes \mathbf{1}\right)$.
- Thus, it suffices to show the following morphism is a levelwise weak $\mathbb{A}^1$-equivalence:

$$\mathrm{Sus}^0_{T_X,\Sigma}\left(\mathbb{A}^1_X \otimes \mathbf{1}\right) \to \mathrm{Sus}^0_{T_X,\Sigma}\left(\mathrm{id}_X \otimes \mathbf{1}\right)$$

- At the level $n \in \mathbb{N}$, this arrow is given by $\mathbb{A}^1 \otimes T_X^{\otimes n} \to T_X^{\otimes n}$. Now the result is clear.

□

5 Construction of the adjunction $f_! : Sh(X) \rightleftarrows Sh(Y) : f^!$

5.1 Thom autoequivalence

[Ayoub,1.5.1, Def.1.5.1, Prop.1.5.2, p.66; Lem.1.5.3, p.67; Th.1.5.7, p.69]

Consider a sequence of S-morphisms

$$X \xrightarrow{s} V \xrightarrow{p} X$$

such that $p \circ s = \mathrm{Id}_X$ and p smooth.

- Define a 1-morphism :

$$\mathbf{Th}(s,p) := p_\sharp \circ s_* : \mathsf{H}(X) \to \mathsf{H}(X)$$

- This 1-morphism $\mathbf{Th}(s,p)$ admits a right adjoint $\mathbf{Th}^{-1}(s,p)$ defined by

$$\mathbf{Th}^{-1}(s,p) := s^! \circ p^* : \mathsf{H}(X) \to \mathsf{H}(X)$$

- By utilising the adjunction between $\mathbf{Th}(\ldots)$ and $\mathbf{Th}^{-1}(\ldots)$ we define by Proposition 1.1.9 a 2-morphism:

$$\phi_{-1}(f) : f^*\mathbf{Th}^{-1}(s,p) \to \mathbf{Th}^{-1}(s',p')f^*$$

  Always by Proposition 1.1.9, the following diamond (of the commutatiity with the unit):

$$\begin{tikzcd}
 & f^*\mathbf{Th}^{-1}(s,p)\mathbf{Th}(s,p) \arrow[dr, "\phi_{-1}(f)"] & \\
f^* \arrow[ur, "\eta"] \arrow[dr, "\eta"'] & & \mathbf{Th}^{-1}(s',p')f^*\mathbf{Th}(s,p) \\
 & \mathbf{Th}^{-1}(s',p')\mathbf{Th}(s',p')f^* \arrow[ur, "\sim", "\phi(f)"'] &
\end{tikzcd}$$

  is commutatie. There also exists an analogous commutative diamond for the counit.

- Let $f : X' \to X$ be a S-morphism. We choose a commutative diagram of cartesian squares:

$$\begin{tikzcd}
X' \arrow[r, "s'"] \arrow[d, "f"'] & V' \arrow[r, "p'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\
X \arrow[r, "s"] & V \arrow[r, "p"] & X
\end{tikzcd}$$

  Then there exists a 2-isomorphism

$$\phi(f) : \mathbf{Th}(s',p')f^* \xrightarrow{\sim} f^*\mathbf{Th}(s,p)$$

  defined by the composition of the planar diagram:

$$\begin{tikzcd}
\mathsf{H}(X') \arrow[r, "s'_*"] & \mathsf{H}(V') \arrow[r, "p'_\sharp"] & \mathsf{H}(X') \\
\mathsf{H}(X) \arrow[u, "f^*"] \arrow[r, "F'"'] \arrow[ur, phantom, "(Ex^*_*)^{-1} \Downarrow"] & \mathsf{H}(V) \arrow[u, "f'^*"] \arrow[r, "p_\sharp"'] \arrow[ur, phantom, "Ex^*_\sharp \Downarrow"] & \mathsf{H}(X) \arrow[u, "f^*"']
\end{tikzcd}$$

[Ayoub, Th.1.5.7, p.69; Cor.1.5.8, p.70]

- The 1-morphisms $\mathbf{Th}(s,p)$ and $\mathbf{Th}^{-1}(s,p)$ are inverse equivalences of each other.
- The 1-morphisms $\mathbf{Th}(s,p)$ (resp. $\mathbf{Th}^{-1}(s,p)$) is therefore called Thom equivalence (resp. inverse Thom equivalence) associated to the section s of the smooth morphism p.
- Suppose X is a quasi-projective S-scheme and $X \xrightarrow{s} V \xrightarrow{p} X$ a sequence of S-morphisms such that $p \circ s = \mathrm{id}_X$ and p smooth. For any object $X' \to X$ of $(\mathbf{Sch}/X)$ one form the diagram of cartesian square:

$$\begin{tikzcd} X' \arrow[r, "s'"] \arrow[d] & V' \arrow[r, "p'"] \arrow[d] & X' \arrow[d] \\ X \arrow[r, "s"] & V \arrow[r, "p"] & X \end{tikzcd}$$

with $V' = V \times_X X'$. Then the pair of families

$$((\mathbf{Th}(s',p'))_{X'\to X}, (\phi(f))_{f:X''\to X'})$$

define an autoequivalence of 2-functor $\mathsf{H}^*_{|(\mathbf{Sch}/X)}$ which is the restriction of the 2-functor H^* to $(\mathbf{Sch}/X)$.

[Ayoub, Th.1.5.9, p.71]

Let $X \xrightarrow{s} V \xrightarrow{p} X$ be a sequence of S-morphisms such that $p \circ s = \mathrm{id}_X$ and p smooth. For any object $X' \to X$ of $(\mathbf{Sch}/X)$ one form the diagram of cartesian squares:

$$\begin{tikzcd} X' \arrow[r, "s'"] \arrow[d] & V' \arrow[r, "p'"] \arrow[d] & X' \arrow[d] \\ X \arrow[r, "s"] & V \arrow[r, "p"] & X \end{tikzcd}$$

For any X-morphism $f : X'' \to X'$, we have the following 2-isomorphisms:

- $\phi(f) : \mathbf{Th}(s'',p'')f^* \xrightarrow{\sim} f^*\mathbf{Th}(s',p')$,
- $\psi(f) : f_*\mathbf{Th}(s'',p'') \xrightarrow{\sim} \mathbf{Th}(s',p')f_*$,
- $\chi(f) : f_\sharp\mathbf{Th}(s'',p'') \xrightarrow{\sim} \mathbf{Th}(s',p')f_\sharp$ if f is smooth,
- $\zeta(i) : \mathbf{Th}(s'',p'')i^! \xrightarrow{\sim} i^!\mathbf{Th}(s',p')$ if $i = f$ and is an immersion.

The family of equivalences $(\mathbf{Th}(s',p'))_{X'\to X}$ provided with the 2-isomorphisms ($\phi(.)$ (resp. $\psi(.), \chi(.)$ and $\xi(.)$) define an autoequialence on he 2-functór H^* resp. H_*; ${}^{\mathrm{Liss}}H_\sharp$ and ${}^{\mathrm{Imm}}H^!$) restricted to X-schemes. Furthermore, these autoequialences are compatible with all the exchange structures construced until now.

[Ayoub, p.72; Prop.1.5.11, p.72; Cor.1.5.13, p.73]

- Suppose given a commutative diagram:

$$\begin{tikzcd} & W \arrow[d, "q"] \arrow[dr, "p\circ q"] & \\ X \arrow[ur, "t"] \arrow[r, "s"'] & V \arrow[r, "p"'] & X \end{tikzcd}$$

with s a closed immersion, p and q smooth and $p \circ s = \mathrm{id}_X$. Then it follows that

$$p \circ q \circ t = p \circ s = \mathrm{id}_X$$

and that t is a closed immersion. We can form the diagram:

$$X \xrightarrow{u=t\times\mathrm{id}} W\times_X \xrightarrow{r=pr_2} X$$

We then have that u is a closed immersion, r is smooth and $r \circ u = \mathrm{id}_X$.
We will construct a 2-isomorphism (of composition):

$$C : \mathbf{Th}(t, p\circ q) \xrightarrow{\sim} \mathbf{Th}(s,p) \circ \mathbf{Th}(u,r)$$

For this we form the diagram of S-schemes :

$$\begin{tikzcd} X \arrow[d, "u"'] \arrow[dr, "t"] & & \\ W\times_V X \arrow[r, "pr_1"] \arrow[d, "r"'] & W \arrow[d, "q"] \arrow[dr, "p\circ q"] & \\ X \arrow[r, "s"'] & V \arrow[r, "p"'] & X \end{tikzcd}$$

and we take the composition of the following planaer diagram:

$$\begin{tikzcd} \bullet \arrow[d, "u_*"'] \arrow[dr, "t_*"] & & \\ \bullet \arrow[u, phantom, ""] \arrow[r, "pr_{1*}"'] \arrow[d, "r_\sharp"'] & \bullet \arrow[l, phantom, "\swarrow c_*" near end, shift left=3] \arrow[d, "q_\sharp"] \arrow[dr, "(p\circ q)_\sharp"] \arrow[dl, phantom, "\swarrow Ex_{*,\sharp}"] & \\ \bullet \arrow[r, "s_*"'] & \bullet \arrow[r, "p_\sharp"'] \arrow[ur, phantom, "\swarrow c_\sharp"] & \bullet \end{tikzcd}$$

The 2-morphism thus obtained is invertible becuase $Ex_{*,\sharp}$ is an isomorphism since q is smooth and s is a closed immersion (see Corollary 1.4.18).

- (This statement completes Theorem 1.5.9):
Under the hypothesis of Theorem 1.5.9, the 2-isomorphisms, which we just built, define an isomorphism of autoequialences between the autoequialence $(\mathbf{Th}(t', p' \circ q'))_{X'\to X}$ nd the composed autoequivalence $(\mathbf{Th}(s', p') \circ \mathbf{Th}(u', r'))_{X'\to X}$, which is for the restrictions of the 2-functorss : $(H^*, H_*, {}^{\mathrm{Liss}}H_\sharp$ and ${}^{\mathrm{Imm}}H^!$ to the category $(\mathbf{Sch}/X)$.

- The Thom equivalences switch between them. More precisely, if $X \xrightarrow{s_1} V_1 \xrightarrow{p_1} X$ and $X \xrightarrow{s_2} V_2 \xrightarrow{p_2} X$ be two sequences as before,then there exists a 2-isomorphism:

$$Cm : \mathbf{Th}(s_1, p_1) \circ \mathbf{Th}(s_2, p_2) \to \mathbf{Th}(s_2, p_2) \circ \mathbf{Th}(s_1, p_1)$$

Furthermore, Cm define an isomorphism of autoequivalences between $(\mathbf{Th}(s'_1, p'_1) \circ \mathbf{Th}(s'_2, p'_2))_{X'\to X}$ and $(\mathbf{Th}(s'_2, p'_2) \circ \mathbf{Th}(s'_1, p'_1))_{X'\to X}$, which is for the restrictions of 2-functors : $H^*, H_*, {}^{\mathrm{Liss}}H^!$ and ${}^{\mathrm{Imm}}H^!$ to the category $(\mathbf{Sch}/X)$.

[Ayoub, Prop.1.5.14, p74; Cor.1.5.15, p.76]

- From a commutative diagram of S-schemes:

$$\begin{tikzcd}
 & U \arrow[d, "m"] \arrow[ddr, "p\circ q\circ m"] & \\
 & W \arrow[d, "q"] \arrow[dr, "p\circ q"'] & \\
X \arrow[uur, "n"] \arrow[ur, "t"] \arrow[r, "s"'] & V \arrow[r, "p"'] & X
\end{tikzcd}$$

form a commutative diagram:

$$\begin{tikzcd}
 & U\times_V X \arrow[d, "x"] \arrow[dr] & \\
X \arrow[ur, "v"] \arrow[r, "u"'] & W\times_V X \arrow[r, "r"'] & X
\end{tikzcd}$$

and the sequence :

$$X \xrightarrow{w} (U\times_V X)\times_{(W\times_V X)} X = U\times_W X \xrightarrow{y} X$$

Then the following square of 2-isomorphisms is commutative:

$$\begin{tikzcd}
\mathbf{Th}(n, p\circ q\circ m) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathbf{Th}(t, p\circ q)\circ\mathbf{Th}(w, y) \arrow[d, "\sim"] \\
\mathbf{Th}(s, p)\circ\mathbf{Th}(v, r\circ x) \arrow[r, "\sim"] & \mathbf{Th}(s, p)\circ\mathbf{Th}(u, r)\circ\mathbf{Th}(w, y)
\end{tikzcd}$$

- Suppose given a commutative diagram:

$$\begin{tikzcd}
 & W \arrow[d, "q"] \arrow[dr, "p\circ q"] & \\
X \arrow[ur, "t"] \arrow[r, "s"'] & V \arrow[r, "p"'] & X
\end{tikzcd}$$

as before and keep the preceeding notations. Let $X \xrightarrow{z} R \xrightarrow{k} X$ be a third such a sequence. Then the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{Th}(t, p\circ q)\circ\mathbf{Th}(z, k) \arrow[rr, "Cm"] \arrow[d, "C"'] & & \mathbf{Th}(z, k)\circ\mathbf{Th}(t, p\circ q) \arrow[d, "C"] \\
\mathbf{Th}(s, p)\circ\mathbf{Th}(u, r)\circ\mathbf{Th}(z, k) \arrow[r, "Cm"] & \mathbf{Th}(s, p)\circ\mathbf{Th}(z, k)\circ\mathbf{Th}(u, r) \arrow[r, "Cm"] & \mathbf{Th}(z, k)\circ\mathbf{Th}(s, p)\circ\mathbf{Th}(u, r)
\end{tikzcd}$$

[Ayoub, Def.1.5.16, Th.1.5.17, Th.1.5.18, p.77]

- Suppose given a commutative diagram:

$$\begin{tikzcd} & W \arrow[dl, "t"'] \arrow[d, "q"] \arrow[dr, "p\circ q"] & \\ X \arrow[r, "s"'] & V \arrow[r, "p"'] & X \end{tikzcd}$$

with s a closed immersion, p and q smooth and $p \circ s = \mathrm{id}_X$. With the preceeding notations, we set C' the composed 2-isomorphism

$$C' : \mathbf{Th}(t, p\circ q) \xrightarrow{C} \mathbf{Th}(s,p) \circ \mathbf{Th}(u,r) \xrightarrow{C_m} \mathbf{Th}(u,r) \circ \mathbf{Th}(s,p)$$

The 2-isomorphism C' is called the <u>2-isomorphism of the change composition</u>. The inverse of the adjoint of C' shall be denoted by C'_{-1}, which is the composite:

$$C'_{-1} : \mathbf{Th}^{-1}(t, p\circ q) \xrightarrow{C_{-1}} \mathbf{Th}^{-1}(u,r) \circ \mathbf{Th}^{-1}(s,p) \xrightarrow{C_{m-1}} \mathbf{Th}^{-1}(s,p) \circ \mathbf{Th}^{-1}(u,r)$$

- Under the hypothesis of Theorem 1.5.9, the 2-isomorphism C' define an isomrophism of autoequivalences between the autoequivalence $(\mathbf{Th}(t', p'\circ q')_{X'\to X}$ and the composed autoequivalence $(\mathbf{Th}(u',r') \circ \mathbf{Th}(s',p'))_{X'\to X}$, which is for the restrictions of the 2-functors : H^*, H_*, ${}^{\mathrm{Liss}}H_\sharp$ and ${}^{\mathrm{Imm}}H^!$ to the category $(\mathbf{Sch}/X)$.
  Furthermore, under the hypothesis of Proposition 1.5.14, the square of 2-isomorphisms:

$$\begin{tikzcd} \mathbf{Th}(n, p\circ q\circ m) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathbf{Th}(w,y) \circ \mathbf{Th}(t, p\circ q) \arrow[d, "\sim"] \\ \mathbf{Th}(v, r\circ x) \circ \mathbf{Th}(s,p) \arrow[r, "\sim"] & \mathbf{Th}(w,y) \circ \mathbf{Th}(u,r) \circ \mathbf{Th}(s,p) \end{tikzcd}$$

- Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of finite dimension. Denote by $p : \mathbb{V}(\mathcal{L}) \to X$ the associated vector bundle. This is the spectrum of the symmetric $\mathcal{O}_X$-algebra $\oplus_{i\geq 0} \mathrm{Sym}^i \mathcal{L}$ associated to $\mathcal{L}$. If $s : X \to \mathbb{V}(\mathcal{L})$ is the zero section, we denote by $\mathbf{Th}(\mathcal{L})$ the equivalanece $\mathbf{Th}(s,p)$.
- When $\mathcal{L} = \mathcal{O}_X$ we also set $\mathbf{Th}(\mathcal{O}_X)A[-2] = A(+1)$. The 1-morphism

$$\mathbf{Th}(\mathcal{O}_X)[-2] : A \to A(+1)$$

is known under the name of <u>Tate twist</u>.
- For an exact sequence of locally free $\mathcal{O}_X$-modules :

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{L} \to 0$$

there exists a 2-isomorphism of composition :

$$C : \mathbf{Th}(\mathcal{M}) \to \mathbf{Th}(\mathcal{N}) \circ \mathbf{Th}(\mathcal{L})$$

and a 2-isomorphism of change composition :

$$C' : \mathbf{Th}(\mathcal{M}) \to \mathbf{Th}(\mathcal{L}) \circ \mathbf{Th}(\mathcal{N})$$

5.2 $f_! \dashv f^!$ in $(\mathbf{Sch}/S)^{\mathrm{Liss}}$ and a cross functor structure on $(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_!, {}^{\mathrm{Liss}}\mathsf{H}^!)$

[Ayoub, ${}^{\mathrm{Liss}}\mathsf{H}^!$, p.78-79]

Definition of $f^!$ For a smooth morphism f, set

$$f^! = \mathbf{Th}(\Omega_f) \circ f^*$$

where Ω_f is the sheaf of relative diffrentials.

Definition of th econnection 2-morphism For a sequence of smooth S-morphisms $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, we take $c^!(f,g)$ by the composite:

$$\begin{tikzcd}
(g \circ f)^! \arrow[d, equal] & & & f^! g^! \arrow[d, equal] \\
\mathbf{Th}(\Omega_{g\circ f})(g \circ f)^* \arrow[r, "c^*(f,g)"] & \mathbf{Th}(\Omega_{g\circ f}) f^* g^* \arrow[r, "C'"] & \mathbf{Th}(\Omega_f)\mathbf{Th}(f^*\Omega_g) f^* g^* \arrow[r, "\phi(f)^{-1}"] & \mathbf{Th}(\Omega_f) f^* \mathbf{Th}(\Omega_g) g^*
\end{tikzcd}$$

with C' the change composition 2-isomorphism (Th.1.5.18) associated to the short exact sequence:

$$0 \to f^*\Omega_g \to \Omega_{g\circ f} \to \Omega_f \to 0$$

Cocycle axiom [DV, 2-functor, Def.2.2] Cosider three composable smooth morphisms:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$$

We must verify that the following diagram is commutative:

$$\begin{tikzcd}
(hgf)^! \arrow[r] \arrow[d] & (gf)^! h^! \arrow[d] \\
f^!(hg)^! \arrow[r] & f^! g^! h^!
\end{tikzcd}$$

[Ayoub, ${}^{\mathrm{Liss}}\mathsf{H}_!$, p.79]

By Prop.1.1.17, there exists a unique global left adjoint ${}^{\mathrm{Liss}}\mathsf{H}_!$ of ${}^{\mathrm{Liss}}\mathsf{H}^!$ such that :

- For any smooth S-morphism f,
$$\mathsf{H}_!(f) = f_! = f_\sharp \mathbf{Th}^{-1}(\Omega_f)$$
- The unit and the counit 2-isomorphisms are respectively::
$$\begin{tikzcd}
\mathrm{id} \arrow[r] & \mathbf{Th}(\Omega_f)\mathbf{Th}^{-1}(\Omega_f) \arrow[d] \\
& \mathbf{Th}(\Omega_f) f^* f_\sharp \mathbf{Th}^{-1}(\Omega_f)
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
f_\sharp \mathbf{Th}^{-1}(\Omega_f)\mathbf{Th}(\Omega_f) f^* \arrow[r] & f_\sharp f^* \arrow[d] \\
& \mathrm{id}
\end{tikzcd} :$$
- The connection 2-isomorphism relative to the sequence $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ is given by the composite of the 2-isomorphisms::

$$\begin{tikzcd}
(g \circ f)_! \arrow[d, equal] & & & g_! f_! \arrow[d, equal] \\
(g \circ f)_\sharp \mathbf{Th}^{-1}(\Omega_{g\circ f}) \arrow[r, "c_\sharp(f,g)"] & g_\sharp f_\sharp \mathbf{Th}^{-1}(\Omega_{g\circ f}) \arrow[r, "C'_{-1}"] & g_\sharp f_\sharp \mathbf{Th}^{-1}(f^*\Omega_g)\mathbf{Th}^{-1}(\Omega_f) \arrow[r] & g_\sharp \mathbf{Th}^{-1}(\Omega_g) f_\sharp \mathbf{Th}^{-1}(\Omega_f)
\end{tikzcd} :$$

[Ayoub, Prop.1.5.19, p.79-81]

There exists a cross functor structure on the quadruplet:

$$\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_!, {}^{\mathrm{Liss}}\mathsf{H}^!\right)$$

such that for any cartesian square of S-schemes (C) :

$$\begin{tikzcd} \bullet \arrow[r, "g'"] \arrow[d, "f'"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

with f smooth, we have:

1. The exchange 2-morphism $Ex_{*,!}(C) : f_! g'_* \to g_* f'_!$ relative to the exchange on $\left(\mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_!\right)$ and associated to the square (C) is given by the composite of planar diagram:

$$\begin{tikzcd} \bullet \arrow[r, "g'_*"] \arrow[d, "\mathbf{Th}^{-1}(\Omega_{f'})"'] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d, "\mathbf{Th}^{-1}(\Omega_f)"] \\ \bullet \arrow[r, "g'_*"] \arrow[d, "f'_\sharp"'] \arrow[dr, phantom, "\swarrow Ex_{*,\sharp}(C)"] & \bullet \arrow[d, "f_\sharp"] \\ \bullet \arrow[r, "g_*"'] & \bullet \end{tikzcd}$$

2. The exchange 2-morphism $Ex^{*,!}(C) : g'^* f^! \to f'^! g^*$ relative to the exchange on $\left(\mathsf{H}^*, {}^{\mathrm{Liss}}\mathsf{H}^!\right)$ and associated to the square (C) is given by the composite of planar diagram:

$$\begin{tikzcd} \bullet \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[l, "g'^*"'] \\ \bullet \arrow[u, "\mathbf{Th}(\Omega_{f'})"] \arrow[dr, phantom, "\swarrow Ex^{*,*}(C)"] & \bullet \arrow[l, "g'^*"'] \arrow[u, "\mathbf{Th}(\Omega_f)"'] \\ \bullet \arrow[u, "f'^*"] & \bullet \arrow[l, "g^*"] \arrow[u, "f^*"'] \end{tikzcd}$$

3. The exchange 2-isomorphism $Ex^!_*(C) : f^! g_* \leftrightarrow g'_* f'^!$ relative to the exchange on $\left(\mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}^!\right)$ and associated to the square (C) is given by the composite of planar diagram:

$$\begin{tikzcd} \bullet \arrow[r, "g'_*"] \arrow[dr, phantom, "\nwarrow"] & \bullet \\ \bullet \arrow[u, "\mathbf{Th}(\Omega_{f'})"] \arrow[r, "g'_*"] \arrow[dr, phantom, "Ex^*_*(C)^{-1}"] & \bullet \arrow[u, "\mathbf{Th}(\Omega_f)"'] \\ \bullet \arrow[u, "f'^*"] \arrow[r, "g_*"'] & \bullet \arrow[u, "f^*"'] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[r, "g'_*"] \arrow[dr, phantom, "\searrow"] & \bullet \\ \bullet \arrow[u, "\mathbf{Th}(\Omega_{f'})"] \arrow[r, "g'_*"] \arrow[dr, phantom, "Ex^*_*(C)"] & \bullet \arrow[u, "\mathbf{Th}(\Omega_f)"'] \\ \bullet \arrow[u, "f'^*"] \arrow[r, "g_*"'] & \bullet \arrow[u, "f^*"'] \end{tikzcd}$$

in the direction of the exchange ($\nwarrow$ or $\searrow$).

4. The exchange 2-isomorphism $Ex^*_!(C) : g^* f_! \leftrightarrow f'_! g'^*$ relative to the exchange on $\left(\mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_!\right)$ and associated to the square (C) is given by the composite of planar diagram:

$$\begin{tikzcd} \bullet \arrow[d, "\mathbf{Th}^{-1}(\Omega_{f'})"'] \arrow[dr, phantom, "\nwarrow"] & \bullet \arrow[l, "g'^*"'] \arrow[d, "\mathbf{Th}^{-1}(\Omega_f)"] \\ \bullet \arrow[d, "f'_\sharp"'] \arrow[dr, phantom, "Ex^*_\sharp(C)^{-1}"] & \bullet \arrow[l, "g'^*"'] \arrow[d, "f_\sharp"] \\ \bullet & \bullet \arrow[l, "g^*"] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[d, "\mathbf{Th}^{-1}(\Omega_{f'})"'] \arrow[dr, phantom, "\searrow"] & \bullet \arrow[l, "g'^*"'] \arrow[d, "\mathbf{Th}^{-1}(\Omega_f)"] \\ \bullet \arrow[d, "f'_\sharp"'] \arrow[dr, phantom, "Ex^*_\sharp(C)"] & \bullet \arrow[l, "g'^*"'] \arrow[d, "f_\sharp"] \\ \bullet & \bullet \arrow[l, "g^*"] \end{tikzcd}$$

in the direction of the exchange ($\nwarrow$ or $\searrow$).

5.3 $f_! \dashv f^!$ in $(\mathbf{Sch}/S)$

[Ayoub, p.145, Prop.1.6.43, p.145; Prop.1.6.44, p.149]

- Starting with two 2-functors:

$$^{\mathrm{Imm}}\mathsf{H}^!; (\mathbf{Sch}/S)^{\mathrm{Imm}} \to \mathfrak{TR}$$
$$^{\mathrm{Liss}}\mathsf{H}^! : (\mathbf{Sch}/S)^{\mathrm{Liss}} \to \mathfrak{TR}$$

we would like to apply Theorem 1.3.1 to construct a contravariant 2-functor:

$$\mathsf{H}^{\underline{!}} : (\mathbf{Sch}/S) \to \mathfrak{TR}$$

and the isomorphisms of 2-functors:

$$^{\mathrm{Imm}}\mathsf{H}^{\underline{!}} \xrightarrow{\sim} {}^{\mathrm{Imm}}\mathsf{H}^!; \quad {}^{\mathrm{Liss}}\mathsf{H}^{\underline{!}} \xrightarrow{\sim} {}^{\mathrm{Liss}}\mathsf{H}^!$$

We have noted as usual $f^{\underline{!}}$ instead of $\mathsf{H}^{\underline{!}}(f)$. To apply Theorem 1.3.1, for any commutative square (C):

$$\begin{tikzcd} Z \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "i"] & Y \end{tikzcd}$$

with f, g smooth and i, k closed immersions, we must:

– construct a 2-isomorphism:

$$a(C) : k^! f^! \xrightarrow{\sim} g^! i^!$$

– and then verify the compatibilities with the compositions of squares.

- To construct $a(C)$, form the commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "s"'] \arrow[rr, bend left, "k"] \arrow[dr, "g"'] & X \times_Y T \arrow[r, "i'"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\ & T \arrow[r, "i"] & Y \end{tikzcd}$$

and we take $a(C)$ by the composite of 2-isomorphisms:

$$k^! f^! \xrightarrow{c^*} s^! t'^! f^! \xrightarrow{(Ex^{!!})^{-1}} s^! f'^! i^! \xrightarrow{\Pi} g^! i^!$$

In other words, $a(C)$ is the composite of the planar diagram:

$$\begin{tikzcd} \mathsf{H}(Z) & \mathsf{H}(X \times_Y T) \arrow[l, "s^!"'] \arrow[dl, phantom, "\Swarrow\Pi"] & \mathsf{H}(X) \arrow[l, "i'^!"'] \arrow[ll, bend right, "k^!"'] \arrow[l, phantom, bend right=20, "\Downarrow c^!"] \\ & \mathsf{H}(T) \arrow[ul, "g^!"] \arrow[u, "f^!"'] & \mathsf{H}(Y) \arrow[l, "i^!"] \arrow[u, "f^!"'] \arrow[ul, phantom, "\Swarrow (Ex^{!,!})^{-1}"] \end{tikzcd}$$

- The compatibilities with the compositions of squares are verified in the following propositions, whose proofs forced Ayoub to draw so many commutative and planar diagrams!

Prop.1.6.43 The 2-isomorphisms $a(C)$ are compatible with the vertical composition of the squares.

Prop.1.6.44 The 2-isomorphisms $a(C)$ are compatible with the horizontal composition of the squares.

[Ayoub, p.152, Lem.1.6.45, p.152; Prop. 1.6.46, p.153]

- So, we are in the condition to apply Theorem 1.3.1. One thus obtain a contravariant 2-functor:

$$\mathsf{H}^{\underline{!}} : (\mathbf{Sch}\,/S) \to \mathfrak{TR}$$

and the isomorphisms of 2-functors:

$${}^{\mathrm{Imm}}\mathsf{H}^{\underline{!}} \xrightarrow{\sim} {}^{\mathrm{Imm}}\mathsf{H}^{!}; \quad {}^{\mathrm{Liss}}\mathsf{H}^{\underline{!}} \xrightarrow{\sim} {}^{\mathrm{Liss}}\mathsf{H}^{!}$$

We have noted as usual $f^{\underline{!}}$ instead of $\mathsf{H}^{\underline{!}}(f)$.

- To construct the 2-functor $\mathsf{H}_{\underline{!}}$, we remark that for any S-morphism f, which admits a factorization

$$f = p \circ s$$

with s a closed immersion and p smooth, the 1-morphism

$$f^{\underline{!}} \xrightarrow{\sim} f^{!} = s^{!}p^{!} = s^{!}\mathbf{Th}\,(\Omega_p)\,p*$$

admits a left adjoint $f_!$ given by:

$$f_! = p_\sharp \mathbf{Th}^{-1}\,(\Omega_p)\,s_*$$

- Replacing $H^{\underline{!}}$ and $H_{\underline{!}}$ by the isomorphic 2-functors, we may suppose the equalities:

$${}^{\mathrm{Liss}}\mathsf{H}^{\underline{!}} = {}^{\mathrm{Liss}}\mathsf{H}^{!}; \quad {}^{\mathrm{Imm}}\mathsf{H}^{\underline{!}} = {}^{\mathrm{Imm}}\mathsf{H}^{!}; \quad {}^{\mathrm{Liss}}\mathsf{H}_{\underline{!}} = {}^{\mathrm{Liss}}\mathsf{H}_{!}; \quad {}^{\mathrm{Imm}}\mathsf{H}_{\underline{!}} = {}^{\mathrm{Imm}}\mathsf{H}_{*}.$$

- What has been shown can be summarized as follows:

[Ayoub, Prop.1.6.46, p.153]

There exists a unique, up to an isomorphism, pair of 2-functors:

$$\mathsf{H}^{!}, \mathsf{H}_{!} : (\mathbf{Sch}\,/S) \to \mathfrak{TR}$$

global adjoint each other ($\mathsf{H}^{!}$ is a right global adjoint of $\mathsf{H}_{!}$) such that:

- $\mathsf{H}^{!}$ extends (in strict sense) the two 2-functors : ${}^{\mathrm{Imm}}\mathsf{H}^{!}$ and ${}^{\mathrm{Liss}}\mathsf{H}^{!}$,
- $\mathsf{H}_{!}$ extends (in strict sense) the two 2-functors : ${}^{\mathrm{Imm}}\mathsf{H}_{!}$ and ${}^{\mathrm{Liss}}\mathsf{H}_{!}$,
- The exchange with respect to the commutatie squre on the pair $({}^{\mathrm{Imm}}\mathsf{H}^{!}, {}^{\mathrm{Liss}}\mathsf{H}^{!})$ constructed in the sub-section 1.6.5, becomes the trivial exchange induced by the connection 2-isomorphism of $\mathsf{H}^{!}$.

5.4 A cross functor structure on $\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}^!\right)$ and proper base change theorem (for a closed immersion)

[Ayoub, Lem.1.4.14, p.61]

(Proper base change theorem (for a closed immersion)) Consider a cartesian square:

$$\begin{tikzcd} T \arrow[r, "g'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

with i a closed immersion. Then the exchange 2-morphism

$$Ex^*_* : g^* i_* \xrightarrow{\sim} i'_* g'^*$$

is invertible.

[Ayoub, Prop.1.4.15, p.62]

(There exists a cross functor structure on the quadruplet:

$$\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}^!\right) \quad)$$

We have a cross functor from $(\mathbf{Sch}\,/S)$ and $(\mathbf{Sch}\,/S)^{\mathrm{Imm}}$ to $\mathfrak{TR}$ with respect to the class of cartesian squares with vertical arrows closed immersions. This cross functor is defined by the data:

- The 2-functor H^* and its right global adjoint H_*,
- The 2-functor ${}^{\mathrm{Imm}}\mathsf{H}^!$ and its left global adjoint ${}^{\mathrm{Imm}}\mathsf{H}_*$,
- The trivial exchange structure on $(\mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}_*)$,
- The exchange structure on $(\mathsf{H}^*, {}^{\mathrm{Imm}}\mathsf{H}^!)$ deduced from the isoexchange of type $\searrow$, inverse of the exchange on $(\mathsf{H}^*, {}^{\mathrm{Imm}}\mathsf{H}_*)$ (with respect to the cartesian squares) and the global adjunction between ${}^{\mathrm{Imm}}\mathsf{H}_*$ and ${}^{\mathrm{Imm}}\mathsf{H}^!$.

For a cartesian square (C) :

$$\begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "i'"'] & \bullet \arrow[d, "i"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

with i a closed immersion, the exchange 2-morphism of the exchange structure on $(\mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}^!)$ is given by $Ex^!_*(C) = {}^a\left(Ex^*_*(C)\right)$. It is therefore the composite:

$$f'_* i'^! \xrightarrow{\eta^!_*(i)} i^! i_* f'_* i'^! \xrightarrow{c_*(f',i)^{-1}} i^!(i \circ f')_* i'^! = i^!(f \circ i')_* i'^! \xrightarrow{c_*(i',f)} i^! f_* i'_* i'^! \xrightarrow{\delta^!_*(i'^!)} i^! f'_*$$

The exchange on $\left(\mathsf{H}^*, {}^{\mathrm{Imm}}\mathsf{H}^!\right)$ is given by the exchange 2-morphism $Ex^{!,*}(C) : f'^* i^! \to i'^! f^*$ equal to the composite:

$$f'^* i^! \xrightarrow{\eta^!_*(i)} i'^! i'_* f'_* i^! \xrightarrow{Ex^*_*(C)^{-1}} i'^! f^* i_* i^! \xrightarrow{\delta^!_*(i)} i'^! f^*$$

5.5 A cross functor structure on $(\mathsf{H}^*, \mathsf{H}_*, \mathsf{H}_!, \mathsf{H}^!)$

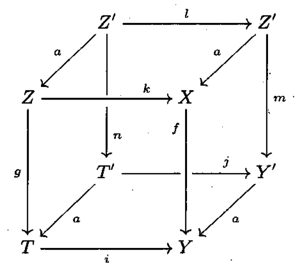
[Ayoub, p.153, Prop.1.6.47, p.153]

- Starting with two exchange structures:
  - $(\mathsf{H}^*, {}^{\mathrm{Imm}}\mathsf{H}^!)$ (see Prop.1.4.15),
  - $(\mathsf{H}^*, {}^{\mathrm{Liss}}\mathsf{H}^!)$ (see Prop.1.5.19),

  we would like to apply Proposition 1.2.7 to construct an exchange on

$$(\mathsf{H}^*, \mathsf{H}^!)$$

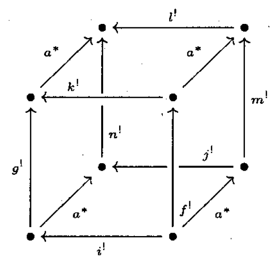
  To apply Proposition 1.2.7, we shall prove the following Proposition 1.6.47:
- Suppose given a commutative diagram of S-schemes:



with i, j, k, l closed immersions, f, g, mn, n smooth morphisms, and such that the four squares having tow parallel ediges worded (by these eight morhipsims) are cartesian. Then, then the diagram of 2-isomorphisms below:

$$\begin{tikzcd}
l^! m^! a^* \arrow[r, "Ex^{!*}"] \arrow[d, "\sim"'] & l^! a^* f^! \arrow[r, "Ex^{!*}"] & a^* k^! f^! \arrow[d, "\sim"] \\
n^! j^! a^* \arrow[r, "Ex^{!*}"'] & n^! a^* i^! \arrow[r, "Ex^{!*}"'] & a^* g^! i^!
\end{tikzcd}$$

is commutative. In other words, the cube:



is commutative.
- Ayoub proved the above claim by drawing many commutative and planar diagrams. In this way, we obtain an exchange structure on

$$(\mathsf{H}^*, \mathsf{H}^!)$$

[Ayoub, p.155; Prop.1.6.48, Lem.1.6.49, p.155]

- Now we would like to prove:

[Ayoub, Prop.1.6.48, p.155]

There exists a unique cross functor structure on the quadruplet $(\mathsf{H}^*, \mathsf{H}_*, \mathsf{H}_!, \mathsf{H}^!)$ which exctends the following two cross functors:
– $\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}_!, {}^{\mathrm{Imm}}\mathsf{H}^!\right)$ of Proposition 1.4.15.
– $\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_!, {}^{\mathrm{Liss}}\mathsf{H}^!\right)$ of Proposition 1.5.19.

by applying Proposition 1.2.14 with

$$(\mathsf{H}^*, \mathsf{H}_*, \mathsf{H}_!, \mathsf{H}^!) = (\mathsf{G}^1, \mathsf{F}_1, \mathsf{F}_2, \mathsf{G}_2)$$

- Actually, this is now easily taken care of as follows:
  - From Prop.1.6.47 and Prop.1.2.7, we established an exchange structure on

$$(\mathsf{H}^*, \mathsf{H}^!).$$

  On the other hand, from Prop.1.6.46, p.153, we obtain a global adjoint pair

$$\mathsf{H}_! \dashv \mathsf{H}^!$$

  From these, we obtain an exchange structure on

$$(\mathsf{H}^*, \mathsf{H}_!) = (\mathsf{G}^1, \mathsf{F}_2)$$

  - We must show this $(\mathsf{H}^*, \mathsf{H}_!)$ is an isoexchange, but this claim follows from the isoexchange properties of the following two exchange structures:
    * The cross functor structure on $\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Imm}}\mathsf{H}_!, {}^{\mathrm{Imm}}\mathsf{H}^!\right)$ of Proposition 1.4.15 gives an exchange structure on

$$(\mathsf{H}^*, {}^{\mathrm{Imm}}\mathsf{H}_!).$$

    * The cross functor structure on $\left(\mathsf{H}^*, \mathsf{H}_*, {}^{\mathrm{Liss}}\mathsf{H}_!, {}^{\mathrm{Liss}}\mathsf{H}^!\right)$ of Proposition 1.5.19 gives an exchange structure on

$$(\mathsf{H}^*, {}^{\mathrm{Liss}}\mathsf{H}_!).$$

  Actually, the former is an isoexchaange by the (proper) base change theorem for a closed immersion (of Lemma 1.4.14, p.61) and the equality

$$i_! = i^*$$

  valid for a closed immersion i.
  The latter is an isoexchange from the axiom 3 and the equality

$$f_! = f_\sharp \mathbf{Th}^{-1}(\Omega_f)$$

  valid for a smooth S-morphism f.

In this way, Proposition 1.6.48 is established. □

The established cross functor structure $(\mathsf{H}^*, \mathsf{H}_*, \mathsf{H}_!, \mathsf{H}^!)$ enjoys the following property:

- The exchange structure on the pair $\left({}^{\mathrm{Imm}}\mathsf{H}_*, \mathsf{H}_!\right)$ induced by restriction of the cross functor constructed in Prop.1.6.48 coincides, module the equality ${}^{\mathrm{Imm}}\mathsf{H}_* = {}^{\mathrm{Imm}}\mathsf{H}_!$. with the exchange induced by restriction of the trivial exchange (i.e. obtained using the connection 2-isomorphism) on the pair $(\mathsf{H}_!, \mathsf{H}_!)$.

6 The proper base change theorem

[Ayoub, Def.1.7.1, p.157]

Given a S-morphism between quasi-projective S-schemes:

$$f : X \to Y$$

(which becomes separated) we form the cartesian square:

$$\begin{tikzcd} X \times_Y X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and denote Δ the diagonal immersion:

$$X \to X \times_Y X$$

The immersion Δ is closed since f is separated. In particular, we have the equality $\Delta_! = \Delta_*$ (in effect : ${}^{\mathrm{Imm}}\mathsf{H}_! = {}^{\mathrm{Imm}}\mathsf{H}_*$).
We then take for $\alpha = \alpha_f$ the composite :

$$\begin{aligned} \alpha_f : f_! = f_! \mathrm{Id}_{X*} &\xrightarrow{c_*} f_! pr_{2*} \Delta_* \xrightarrow{Ex_{*!}} f_* pr_{1!} \Delta_* \\ = f_* pr_{1!} \Delta_! &\xrightarrow{(c_!)^{-1}} f_* \mathrm{Id}_{X!} = f_* \end{aligned}$$

In the language of the plane diagram, α_f is the composite:

$$\begin{tikzcd} \mathsf{H}(X) \arrow[dr] \arrow[drr, equal] \arrow[ddr, equal] & & \\ & \mathsf{H}(X \times_Y X) \arrow[r, "pr_{2*}"'] \arrow[d, "pr_{1!}"] & \mathsf{H}(X) \arrow[d, "f_!"'] \\ & \mathsf{H}(X) \arrow[r, "f_*"'] & \mathsf{H}(Y) \end{tikzcd}$$

[Ayoub, Th.1.7.9, p.167]

Let X be a quasi-projective S-scheme. Denote by p_n the canonical projection $\mathbb{P}^n_X \to X$. Then, the 2-morphism

$$\alpha_{p_n} : p_{n!} \to p_{n*}$$

is invertible.

[Ayoub, Cor.1.7.13, Cor.1.7.14, p.173]

Let p is the projection $\mathbb{P}^n_X \to X$.

- The 2-morphism
$$\alpha_p : p_! p^* \to p_* p^*$$
is invertible.
- The 2-morphism
$$\alpha_p : p_! p^! \to p_* p^!$$
is invertible.

[Ayoub, Th.1.7.17, Cor.1.7.18, p.177]

- Let $f : X \to Y$ be a projective S-morphism. The 2-morphism

$$\alpha_f : f_! \to f_*$$

is invertible.

- Suppose given a cartesian square of S-schemes:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

with f projective. Then the exchange 2-morphism

$$Ex^*_* : g^* f_* \to f'_* g'^*$$

is invertible.

[CD, Prop.2.3.11, p.34]

Consider a morphism of complete triangulated $\mathcal{P}$-fibred categories over $\mathcal{S}$:

$$\phi^* : \mathcal{F} \to \mathcal{F}'$$

- For any morphism $f : Y \to X$ there is an exchange transformation [CD,1.2.5.1]:

$$Ex(\phi^*, f_*) : \phi_X^* f_* \to f_* \phi_Y^*$$

- If $\mathcal{F}$ and $\mathcal{F}'$ satisfy the support axiom and f is separated of finite type, another exchange transformation is constructed [CD, Prop.2.2.11]:

$$Ex(\phi^*, f_!) : \phi_X^* f_! \to f_! \phi_Y^*$$

- These exhange transformations enjoy the following properties:
  - Let $i : Z \to X$ be a closed immersion such that $\mathcal{F}$ and $\mathcal{F}'$ satisfy property (Loc_i). Then the exchange

$$Ex(\phi^*, i_*) : \phi_X^* i_* \to i_* \phi_Z^*$$

    is an isomorphism.
  - Assume $\mathcal{F}$ and $\mathcal{F}'$ satisfy property (Loc). Then the following conditions are equivalent:

    (i) For any integer $n > 0$ and any scheme X in $\mathcal{S}$, the exchange $Ex(\phi^*, p_{n*})$ is an isomorphism where $p_n : \mathbb{P}_X^n \to X$ is the canonical projection.

    (ii) For any proper morphism $f : Y \to X$, the exchange $Ex(\phi^*, f_*)$ is an isomorphism.
  - Assume $\mathcal{F}$ and $\mathcal{F}'$ satisfy properties (Loc) and $(Supp)$. Then conditions (i) and (ii) above are equivalent to the following one:

  (iii) For any separated morphism $f : Y \to X$ of finite type, the exchange $Ex(\phi^*, f_!)$ is an isomorphism.

References

[A] Joseph Ayoub, Les six opérations de Grothendieck et cycles évanescents dans le monde motivique I, II. Astérisque, volumes 314, 315 (2008). Société Mathématique de France. (Available on his web page)

[A2] Joseph Ayoub, Note sur les oprations de Grothendieck et la ralisation de Betti. Journal of the Inst. of Math. Jussieu (2010) 9(2), p. 225263.

[C] Denis-Charles Cisinski, Descente par éclatements en K-théorie invariante par homotopie, Ann. of Math. (2) 177 (2013), no. 2, 425-448.

[CD1] Denis-Charles Cisinski, Frédéric Déglise, Triangulated Categories of Mixed Motives. A draft of which available at **http://perso.enslyon.fr/frederic.deglise/motifs.html**

[CD2] Denis-Charles Cisinski, Frédéric Déglise, Integral mixed motives in equal characteristic, Doc. Math. 2015, Extra vol.: Alexander S. Merkurjev's sixtieth birthday, 145-194.

[DV] Pierre Deligne, Voevodsky ' s lectures on cross functors, Fall 2001, updated January 17 2002,

[FGA] Fundamental Algebraic Geometry. Grothendieck's FGA Explained, Barbara Fantechi, Lothar Gottsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, Mathematical Surveys and Monographs, Vol. 123, American Mathematical Society, 2005.

[Hi] Philip S. Hirschhorn, Model categories and their localizations. Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI, 2003. xvi+457 pp.

[Ho1] Mark Hovey. — Model categories. Mathematical Surveys and Monographs 63, AMS, Providence, RI, (1999), x + 209 pp.

[Ho2] Mark Hovey. —Spectra and symmetric spectra in general model categories. Journal of Pure and Applied Algebra, volume 165, (2001), pp. 63-127.

[HSS] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. J. Amer. Math. Soc., 13(1):149-208, 2000.

[Hoy1] Marc Hoyois, The six operations in equivariant motivic homotopy theory, arXiv:1509.02145.

[Hoy2] Marc Hoyois, Equivariant classifying spaces and cdh descent for the homotopy K-theory of tame stacks, arXiv:1604.06410.

[I] Birger Iversen, Cohomology of Sheaves, Springer, 1986.

[IKM] Ryomei Iwasa, Wataru Kai, and Hiroyasu Miyazaki, Haramura 2016 Program (final version), August 31, 2016. A draft prepared for the workshop: **http://www.lcv.ne.jp/ smaki/ja/yatugatakeWS/index.html**

[J] J.F. Jardine, Motivic symmetric spectra, Documenta Mathematica, Band 5, (2000) pp. 445-552.

[KS] Masaki Kashiwara, Pierre Schapira, Sheaves on manifolds, Grundlehren der Mathematischen Wissenschaften, 292, Springer-Verlag, Berlin, Corrected Second Printing 1994, x+512 pp.

[K] Shane Kelly, Some observations about motivic tensor triangulated geometry over a finite field, to appear in Surveys around Ohkawa's theorem on Bousfield classes,

[L] Marc Levine, Mixed Motives, Mathematical Surveys and Monographs Volume 57, American Mathematical Society, 1998,

[LRV] Wolfgang Lueck, Holger Reich, Marco Varisco, Commuting homotopy limits and smash products, K-Theory 30-2 (2003) 137-165

[Ma] G. Maltsiniotis, Introduction la thorie des drivateurs. Preprint available at **http ://www.math.jussieu.fr/ maltsin/**

[Mar] Matilde Marcolli, Feynman Motives, World Scientific, 2010,

[Mi] Norihiko Minami, A topologist's introduction to the motivic homotopy theory for transformation group theorists - 1, RIMS Kôkyûroku Bessatsu B39 (2013), 063-107.

[Mil] J. S. Milne, Etale Cohomology, 1980, Princeton University Press.

[MV] Fabien Morel and Vladimir Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, Inst. Hautes Études Sci. Publ. Math. 90 (1999), 45-143 (2001).

[N] Amnon Neeman, Triangulated categories, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[SGA 4] Théorie des topos et cohomologie étale des schémas, séminaire dirigpar M. Artin, A. Grothendieck et J.-L. Verdier, Lecture Notes in Mathematics 269, 270, 305, Springer-Verlag, 1972-73.

[SGA 4+1/2] Cohomologie étale, par P. Deligne, Lecture Notes in Mathematics 569, Springer Verlag, 1977.

[SGA 6] Théorie des intersections et théoréme de Riemann-Roch, par P. Berthelot, A. Grothendieck et L. Illusie, Lecture Notes in Mathematics 225, 1971.

[Toric] Toric topology. Papers from the International Conference held at Osaka City University, Osaka, May 28June 3, 2006. Edited by Megumi Harada, Yael Karshon, Mikiya Masuda and Taras Panov. Contemporary Mathematics, 460. American Mathematical Society, Providence, RI, 2008. xviii+401 pp.